TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 308, Number 2, August 1988

SOME P.V.-EQUIVALENCES AND A CLASSIFICATION OF 2-SIMPLE PREHOMOGENEOUS VECTOR SPACES OF TYPE II

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Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

ABSTRACT. A classification of 2-simple prehomogeneous vector spaces is completed by using some P.V.-equivalences together with [3]. Some part is very different from the previous classification of the irreducible or simple cases [1, 2], and some new method is necessary. This result shows the difficult point of a classification problem of reductive prehomogeneous vector spaces.

Introduction. In M. Sato and T. Kimura [1], all irreducible prehomogeneous vector spaces (abbreviated irreducible P.V.'s) are classified up to castling-equivalence, and it is proved that any irreducible P.V. is castling-equivalent to a 2-simple P.V. (or to $(SL(m) \times SL(m) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ with $m = 2, 3$). Therefore, as a step to a classification of the general case, it is natural to classify all nonirreducible 2-simple P.V.'s. In this paper, we shall classify all 2-simple P.V.'s of type II and give the complete list of them up to strong equivalence in the sense of Definition 4, p. 36 in [1]. However we need various P.V.-equivalences (cf. Propositions 1.7; 1.12; 1.34; 1.36; 1.37, Theorem 1.16 and its Corollary; Corollary 1.26, Definition 1.31, Remark 1.32) to carry out the classification, and hence, first we shall prove them. We also investigate the regularity in some cases. Since all 2-simple P.V.'s of type I are already classified in [3], this completes the classification of all 2-simple P.V.'s. Here we say that (G, ρ, V) (or simply (G, ρ)) is a 2-simple P.V. if it is a prehomogeneous vector space of the following form: $G = GL(1)^{k+s+t} \times G_1 \times G_2$ where G_1 and G_2 are simple algebraic groups, $\rho =$ the composition of the scalar multiplications $GL(1)^{k+s+t}$ on each irreducible component and the representation $\rho' = \rho_1 \otimes \rho_1' + \cdots + \rho_k \otimes \rho_k' + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$ of the group $G_1 \times G_2$ where ρ_i, σ_j (resp. ρ_i', τ_j) are nontrivial irreducible representations of G_1 (resp. G_2). We write (G, ρ', V) instead of (G, ρ, V) for simplicity. To avoid confusion, we write $+$ instead of $\oplus$. Clearly each $(GL(1) \times G_1 \times G_2,\ \rho_i \otimes \rho_i')$ $(i = 1, \ldots, k)$ is an irreducible 2-simple P.V. If at least one of them is a nontrivial P.V. (see Definition 1.1), we call it a 2-simple P.V. of type I. On the other hand, if all of them are trivial P.V.'s, we call it a 2-simple P.V. of type II. In this case, we have $G_2 = SL(n)$,

Received by the editors March 3, 1986 and, in revised form, July 30, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11R20, 11R29; Secondary 11N25, 11R18.

$\rho_i' = \Lambda_1^{(*)}$ (i.e., Λ_1 or Λ_1^*) and $2 \leqslant \deg \rho_i \leqslant n$ for $i = 1, \dots, k$. Here Λ_1 stands for the standard representation of $SL(n)$ on K^n and Λ_1^* its dual. In this paper, we shall consider everything over an algebraically closed field K of characteristic zero.

The work in §§1 to 3 was done essentially around 1982. The crucial part for a classification of 2-simple P.V.'s is §4, in particular §4.2, where we cannot use the previous results in [**1**, **2**], and some new idea was necessary. Namely the classification problem of 2-simple P.V.'s has been reduced to the problem, which was open since 1982, to determine when

$$\Big(GL(1)^{k+s+t} \times SL(m) \times SL(n),\ (\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s}) \otimes 1 + (\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}) + 1 \otimes (\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t})\Big),$$

with $km \gneqq n \gneqq m \geqslant 2$ and $t \geqslant 1$, has a Zariski-dense orbit. The solution is given by Theorem 4.13. The key point is to investigate a castling transformation in detail by Lemmas 4.7–4.9. Note that scalar multiplications have a delicate role in §4. For example,

$$\Big(GL(1)^{k+s+t} \times SL(m) \times SL(n),\ (\Lambda_1^* + \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s-1}) \otimes 1 + (\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}) + 1 \otimes (\Lambda_1^* + \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-1})\Big)$$
$$(s \geqslant 1, t \geqslant 1, km \gneqq n \gneqq m)$$

is a P.V. if and only if

$$\Big(GL(m-1) \times GL(n-1),\ (\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k+s-2}) \otimes 1 + (\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}) + 1 \otimes (\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k+t-2})\Big)$$

is a P.V. (see Theorem 4.18). These results show us the difficulty of the classification of reductive P.V.'s, or even 3-simple P.V.'s. The crucial point will be to classify the case $(GL(1)^s \times SL(m_1) \times \cdots \times SL(m_k),\ \sum_{t=1}^{s}(\rho_1^t \otimes \cdots \otimes \rho_k^t))$ when any $\rho_i^t = \Lambda_1, \Lambda_1^*$, or 1 ($k \geqslant 3$). Even the prehomogeneity of

$$((GL(n_1) \times \cdots \times GL(n_k)) \times GL(n), (\Lambda_1 + \cdots + \Lambda_1) \otimes \Lambda_1)$$

is not completely known for $k \geqslant 6$.

Now, we shall explain about §§1–3. In §1, we investigate the prehomogeneity of $(G \times GL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*)$. Some people already worked on this subject (see Remark 1.38). We also give a proof of invariance of regularity under a castling transformation without assuming the reductivity of the groups. Then we give some

P.V.-equivalences in Proposition 1.34, obtained from Yasukura's remark (Proposition 1.33). In §2, this will be used a lot, together with the results in [**1**, **2**], to classify the case when $(GL(n), \tau)$ $(\tau \neq \Lambda_1, \Lambda_1^*)$ is one of the irreducible components. Although §§2 and 3 have been done before (unpublished), by using the P.V.-equivalence in Proposition 1.36 (resp. Lemma 3.7), we were able to make §2 (resp. §3) much shorter than before. We would like to express our hearty thanks to the referee for giving a counterexample to our previous conjecture (see Remark 1.24).

1. Some P.V.-equivalences and regularity. In this section, we do not assume the reductivity of the groups. First of all, we shall review relative invariants of a P.V. (See §4 in [**1**].) Let (G, ρ, V) be a P.V. with *the singular set* S, i.e., the proper Zariski-closed subset of V such that $V - S$ consists of a single G-orbit. A point x in $V - S$ is called *a generic point* and the isotropy subgroup $G_x = \{g \in G;\ \rho(g)x = x\}$ at a generic point x is called *a generic isotropy subgroup*. Let $S_1, \ldots, S_l$ be the irreducible components of S of codimension one. Then there exist irreducible polynomials $f_j(x)$ and characters χ_i: $G \to K^\times$ satisfying $S_i = \{x \in V;\ f_i(x) = 0\}$ and $f_i(\rho(g)x) = \chi_i(g) f_i(x)$ for all $g \in G$ and $x \in V$ $(i = 1, \ldots, l)$. In general, a nonzero rational function $f(x)$ on V is called *a relative invariant* of (G, ρ, V) if there exists a character χ: $G \to K^\times$ satisfying $f(\rho(g)x) = \chi(g)f(x)$ for all $g \in G$ and $x \in V - S$. It is known that such f can be written uniquely as $f(x) = cf_1(x)^{m_1}, \ldots, f_l(x)^{m_l}$ with $c \in K^\times$ and $(m_1, \ldots, m_l) \in Z^l$, and we say that $f_1(x), \ldots, f_l(x)$ are *basic relative invariants*.

Now put $(G)_1 = [G, G] \cdot G_x$ where G_x is a generic isotropy subgroup. It does not depend on a choice of a generic point. Then the character group of $G/(G)_1$ is a free abelian group of rank l.

Now if there exists a relative invariant $f(x)$ of (G, ρ, V) such that its Hessian $H_f(x) = \det(\partial^2 f/\partial x_i \partial x_j)$ is not identically zero, we say that (G, ρ, V) is *a regular P.V.* It is equivalent to say that the map $\operatorname{grad}\log f$: $V - S \to V^*$ is dominant for some relative invariant $f(x)$. Moreover, when G is reductive, (G, ρ, V) is regular if and only if a generic isotropy subgroup is reductive, and it is so if and only if the singular set S is a hypersurface (pp. 70–73 in [**1**]).

DEFINITION 1.1. Let ρ: $H \to GL(V)$ be any finite-dimensional representation of any group H. Then, a triplet $(H \times GL(n),\ \rho \otimes \Lambda_1, V \otimes V(n))$ is a P.V. for all n satisfying $n \geqslant \deg \rho = \dim V$. We call such a triplet *a trivial P.V.*

LEMMA 1.2. *Let (G, ρ, V) be a regular P.V. and (G', ρ', V) a P.V. satisfying $\rho'(G') \subset \rho(G)$ $(\subset GL(V))$. Then (G', ρ', V) is also a regular P.V.*

PROOF. By definition, there exists a relative invariant $f(x)$ of (G, ρ, V) such that its Hessian is not identically zero. Since $f(x)$ is also a relative invariant of (G', ρ', V), we have our result. Q.E.D.

PROPOSITION 1.3. *A trivial P.V. $(H \times GL(n), \rho \otimes \Lambda_1, V \otimes V(n))$ $(n \geqslant \dim V = \deg \rho)$ is regular if and only if $n = \deg \rho$.*

PROOF. Assume $n = \deg \rho$. Since $(GL(n) \times GL(n),\ \Lambda_1 \otimes \Lambda_1)$ is a regular P.V., $(H \times GL(n),\ \rho \otimes \Lambda_1)$ is a regular P.V. by Lemma 1.2. Actually, we have grad log det $X = {}^tX^{-1}$ for $X \in V - S$ (see Lemma 1.21.). Now assume that $n \gneq \deg\rho$ and $(H \times GL(n),\ \rho \otimes \Lambda_1)$ is regular. Then again by Lemma 1.2., a trivial P.V.

$$(\{1\} \times GL(n), (\overbrace{1 + \cdots + 1}^{m}) \otimes \Lambda_1, V \otimes V(n)) \cong (GL(n), \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{m}) \qquad (m = \deg\rho \lneq n)$$

is regular, which is a contradiction. Q.E.D.

DEFINITION 1.4. We call $(G, \rho, V) = \oplus_{i=1}^{l}(G_i, \rho_i, V_i)$ *the direct sum* of the triplets (G_i, ρ_i, V_i) $(i = 1, \ldots, l)$ when $G = G_1 \times \cdots \times G_l$, $\rho = \rho_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \rho_l$, $V = V_1 \oplus \cdots \oplus V_l$. Then, it is clear that $\oplus_{i=1}^{l}(G_i, \rho_i, V_i)$ is a P.V. if and only if all triplets (G_i, ρ_i, V_i) $(i = 1, \ldots, l)$ are P.V.'s. In particular, if (G_1, ρ_1, V_1) and (G_2, ρ_2, V_2) are simple P.V.'s, we obtain a 2-simple P.V. $\oplus_{i=1}^{2}(G_i, \rho_i, V_i)$ which is called *a decomposable 2-simple P.V.* Here a P.V. (G, ρ, V) is called a simple P.V. if $G = GL(1)^l \times G_s$ and ρ is the composition of a rational representation $\rho' = \rho_1 \oplus \cdots \oplus \rho_l$ of a simple algebraic group G_s and the scalar multiplications $GL(1)^l$ on each irreducible component V_i $(V = V_1 \oplus \cdots \oplus V_l)$ where ρ_i is an irreducible representation of G_s on V_i $(i = 1, \ldots, l)$. Such simple P.V.'s are already classified (see §5.1).

PROPOSITION 1.5. *The direct sum* $\oplus_{i=1}^{l}(G_i, \rho_i, V_i)$ *is regular if and only if* (G_i, ρ_i, V_i) *are regular P.V.'s for all* $i = 1, \ldots, l$.

PROOF. Let S_i be the singular set of (G_i, ρ_i, V_i) for $i = 1, \ldots, l$. Then the singular set of $\oplus_{i=1}^{l}(G_i, \rho_i, V_i)$ is $\bigcup_{i=1}^{l} V_1 \oplus \cdots \oplus S_i \oplus \cdots \oplus V_l$ and hence any relative invariant $f(x)$ of $\oplus_{i=1}^{l}(G_i, \rho_i, V_i)$ is of the form $f(x) = f_1(x_1) \cdots f_l(x_l)$ for $x = (x_1, \ldots, x_l) \in V_1 \oplus \cdots \oplus V_l$ where $f_i(x_i)$ is a relative invariant of (G_i, ρ_i, V_i) for $i = 1, \ldots, l$. Since $\operatorname{grad}\log f(x) = \sum_{i=1}^{l} \operatorname{grad}\log f_i(x_i)$, we have our result. Q.E.D.

From now on, we shall deal only with indecomposable P.V.'s., i.e., $k \geqslant 1$ in the Introduction.

DEFINITION 1.6. A triplet $(\tilde{G}, \tilde{\rho}, \tilde{V})$ is called *a generalized direct sum* of triplets (G_i, ρ_i, V_i) $(i = 1, \ldots, l)$ if $\tilde{G} = G_1 \times \cdots \times G_l \times GL(n)$,

$$\tilde{\rho} = (\rho_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \rho_l) \otimes 1 + \rho \otimes \Lambda_1$$

and

$$\tilde{V} = V_1 + \cdots + V_l + V' \otimes V(n)$$

where ρ: $G_1 \times \cdots \times G_l \to GL(V')$ is any finite-dimensional representation and n is any natural number satisfying $n \geqslant \dim V'$. If $\dim V' = 0$, it is a usual direct sum.

PROPOSITION 1.7. *A generalized direct sum* $(\tilde{G}, \tilde{\rho}, \tilde{V})$ *of* (G_i, ρ_i, V_i) $(i = 1, \ldots, l)$ *is a P.V. if and only if all triplets* (G_i, ρ_i, V_i) $(i = 1, \ldots, l)$ *are P.V.'s.*

PROOF. Assume that all (G_i, ρ_i, V_i) $(i = 1, \ldots, l)$ are P.V.'s. Let H be a generic isotropy subgroup of $\oplus_{i=1}^{l}(G_i, \rho_i, V_i)$. Then the triplet $(\tilde{G}, \tilde{\rho}, \tilde{V})$ is a P.V. if and only if $(H \times GL(n),\ \rho|_H \otimes \Lambda_1,\ V' \otimes V(n))$ is a P.V., and the latter is actually a trivial P.V. The converse is obvious. Q.E.D.

LEMMA 1.8. *Let* $(H \times GL(n),\ \rho \otimes \Lambda_1,\ V(m) \otimes V(n))$ *be any trivial P.V. with* $m \leqslant n$. *If we identify* $V = V(m) \otimes V(n)$ *with the totality of* $m \times n$ *matrices* $M(m, n)$, *then its singular set* S *is given by* $S = \{X \in M(m, n);\ \mathrm{rank}\ X \leqslant m - 1\}$, *and* $V - S$ *consists of a single* $GL(n)$*-orbit.*

PROOF. When

$$(H, \rho, V(m)) = (GL(m), \Lambda_1, V(m))$$

or

$$(H, \rho, V(m)) = \Big(\{1\}, \overbrace{1 + \cdots + 1}^{m}, V(m)\Big),$$

the set $\{X \in M(m, n);\ \mathrm{rank}\ X = m\}$ consists of a single $H \times GL(n)$-orbit, and hence, this is so for any $(H, \rho, V(m))$ since $\{1\} \subset \rho(H) \subset GL(m)$. Thus we have $S = \{X \in M(m, n);\ \mathrm{rank}\ X \leqslant m - 1\}$. Q.E.D.

PROPOSITION 1.9. *A generalized direct sum* $(\tilde{G}, \tilde{\rho}, \tilde{V})$ *of* (G_i, ρ_i, V_i) $(i = 1, \ldots, l)$ *is a regular P.V. if and only if each triplet* (G_i, ρ_i, V_i) $(i = 1, \ldots, l)$ *is a regular P.V. and* $n = \dim V'$.

PROOF. Let S (resp. $S', \tilde{S}$) be the singular set of $\oplus_{i=1}^{l}(G_i, \rho_i, V_i)$ (resp. $(\tilde{G}, \rho \otimes \Lambda_1, V' \otimes V(n))$, $(\tilde{G}, \tilde{\rho}, \tilde{V})$). Then

$$\{(x, x') \in (V_1 \oplus \cdots \oplus V_l) + V' \otimes V(n);$$
$$x \in (V_1 \oplus \cdots \oplus V_l) - S, x' \in V' \otimes V(n) - S'\}$$

consists of a single $\tilde{G}$-orbit by Lemma 1.8.

Hence we have $\tilde{S} = S \oplus (V' \otimes V(n)) \cup V_1 \oplus \cdots \oplus V_l \oplus S'$. From here the proof goes similarly as the proof of Proposition 1.5. by using Proposition 1.3. Q.E.D.

DEFINITION 1.10. From any simple P.V. (G, ρ, V), we obtain a 2-simple P.V. $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V + V' \otimes V(n))$ where $\sigma\colon G \to GL(V')$ is any finite-dimensional rational representation and n is any natural number satisfying $n \geqslant \dim V'$. We call such a triplet *a 2-simple P.V. of trivial type*. By Proposition 1.9, it is regular if and only if $n = \dim V'$ and (G, ρ, V) is regular.

LEMMA 1.11. $(GL(1) \times G,\ \rho_1 \otimes \rho_2,\ V(m_1) \otimes V(m_2))$ *with* $m_1 \geqslant m_2 \geqslant 2$ *is a non-P.V., where* ρ_1 *and* ρ_2 *are nontrivial irreducible representations of a simple algebraic group* G.

PROOF. Since $\rho_2(G) \subset SL(m_2)$, we have $\dim GL(1) \times G \leqslant m_2^2 \leqslant m_1 m_2 = \deg \rho_1 \otimes \rho_2$. On the other hand, if it is a P.V., then $\dim GL(1) \times G \geqslant \deg \rho_1 \otimes \rho_2$, and hence $m_1 = m_2$ $(= m)$, $G = SL(m)$ and $\rho_1 \otimes \rho_2 = \Lambda_1 \otimes \Lambda_1^{(*)}$. However $f(X) = \det(X + {}^tX)/\det X$ (resp. $(\mathrm{Tr}\, X)^m/\det X$) is a nonconstant absolute invariant for $X \in M(m) = V(m_1) \otimes V(m_2)$ when $\rho_1 \otimes \rho_2 = \Lambda_1 \otimes \Lambda_1$ (resp. $\Lambda_1 \otimes \Lambda_1^*$) and so it cannot be a P.V. Q.E.D.

PROPOSITION 1.12. *Let G be a simple algebraic group and consider a P.V.*

$$\big(GL(1)^l \times G \times SL(n),$$
$$(\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)\big)$$

with $2 \leqslant \deg \rho_i \leqslant n$ $(i = 1, \dots, k)$, $l = k + s + t$ *where* ρ_i, σ_i (*resp.* τ_j) *are nontrivial irreducible representations of G* (*resp.* $SL(n)$). *If* $\deg \rho_i = n$ *for some* $i = 1, \dots, k$, *then we have* $k = 1$. *A triplet*

$$\big(GL(1)^{1+s+t} \times G \times SL(n), \rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t),$$
$$M(n) + V(\deg(\sigma_1 + \cdots + \sigma_s)) + V(\deg(\tau_1 + \cdots + \tau_t))\big)$$

with $\deg \rho_1 = n$, *is a P.V. if and only if a triplet*

$$\big(GL(1)^{s+t} \times G, (\sigma_1 + \cdots + \sigma_s) + (\tau_1 + \cdots + \tau_t) \cdot \rho_1^*, V_1 + V_2\big)$$

is a simple P.V. where $\dim V_1 = \deg(\sigma_1 + \cdots + \sigma_s)$ *and* $\dim V_2 = \deg(\tau_1 + \cdots + \tau_t)$.

PROOF. It is clear that $(GL(1)^2 \times G \times SL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1)$ with $\deg \rho_1 = n$ is a P.V. if and only if $(GL(1) \times G, \rho_2 \otimes \rho_1^*)$ is a P.V. By Lemma 1.11, the latter cannot be a P.V., and we have $k = 1$. The last statement is obvious. Q.E.D.

REMARK 1.13. The P.V.-equivalence in Proposition 1.12 is an example of isotropic P.V.-equivalence (see Definition 1.31). Assume that the latter triplet is a simple P.V. If $\tau_1 + \cdots + \tau_t = 0$, then the former triplet is a 2-simple P.V. of trivial type (Definition 1.10). Even if $\tau_1 + \cdots + \tau_t \neq 0$, we still call it a 2-simple P.V. of trivial type.

Now let us consider a triplet

$$\big(G \times GL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, V(m_1) \otimes V(n) + V(m_2) \otimes V(n)^*\big).$$

Without loss of generality, we may assume $m_1 \geqslant m_2$. For simplicity, put $G_n = G \times GL(n)$, $\sigma_n = \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*$, and $W_n = V(m_1) \otimes V(n) + V(m_2) \otimes V(n)^*$. We identify $V(m_1) \otimes V(n)$ (resp. $V(m_2) \otimes V(n)^*$, $V(m_1) \otimes V(m_2)$) with $M(m_1, n)$ (resp. $M(m_2, n)$, $M(m_1, m_2)$).

THEOREM 1.14. *Assume that* (G_n, σ_n, W_n) *is a P.V. with* $n \geqslant m_2$. *Then a triplet* $(G, \rho_1 \otimes \rho_2, V(m_1) \otimes V(m_2))$ *must be a P.V.*

PROOF. Define a map $f\colon W_n = M(m_1, n) \oplus M(m_2, n) \to V(m_1) \otimes V(m_2) = M(m_1, m_2)$ by $f(X, Y) = X{}^tY$. Then we have $f((Z\,|\,0), (I_{m_2}\,|\,0)) = Z$ for any $Z \in M(m_1, m_2)$ and hence the map f is surjective. Since $f(\rho_1(g)X{}^th, \rho_2(g)Yh^{-1}) = \rho_1(g)(X{}^tY){}^t\rho_2(g)$ for all $(g, h) \in G_n = G \times GL(n)$, the map f is $G \times GL(n)$-equivariant, and hence, in view of Lemma 5, p. 36 in **[1]**, we obtain our assertion. Q.E.D.

PROPOSITION 1.15. *Let* ρ_1 *and* ρ_2 (*resp.* ρ) *be irreducible representations of a simple algebraic group* G_1 (*resp.* G_2) *satisfying* $\min\{\deg \rho, \deg \rho_1\} \geqslant \deg \rho_2 \geqslant 2$. *Then* $(GL(1)^2 \times G_1 \times G_2, \rho_1 \otimes \rho + \rho_2 \otimes \rho^*)$ *is not a P.V.*

PROOF. If it is a P.V., then $(GL(1)^2 \times G_1 \times GL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*)$ is also a P.V. for $n = \deg \rho$. By Theorem 1.14, $(GL(1) \times G_1, \rho_1 \otimes \rho_2)$ must be a P.V. This is a contradiction by Lemma 1.11. Q.E.D.

THEOREM 1.16. *If a triplet $(G, \rho_1 \otimes \rho_2, V(m_1) \otimes V(m_2))$ is a P.V., then a triplet (G_n, σ_n, W_n) is also a P.V. for any $n \geqslant m_1$. Moreover, if $Z \in M(m_1, m_2) = V(m_1) \otimes V(m_2)$ is a generic point, then $x = ((I_{m_1} | 0), ({}^tZ | 0)) \in W_n = M(m_1, n) \oplus M(m_2, n)$ is a generic point of (G_n, σ_n, W_n).*

PROOF. It is enough to show the last statement. Let $\mathfrak{G}$ (resp. $\mathfrak{G}_n$) be the Lie algebra of G (resp. G_n). Then the isotropy subalgebra $(\mathfrak{G}_n)_x$ of $\mathfrak{G}_n$ at $X = (I_{m_1} | 0)$ is given by

$$(\mathfrak{G}_n)_x = \left\{ \left(A, \left(\begin{array}{c|c} -{}^t d\rho_1(A) & B \\ \hline 0 & C \end{array} \right) \right); A \in \mathfrak{G},\ B \in M(m_1, n - m_1), \right.$$

$$\left. C \in M(n - m_1) \right\}$$

and it acts on $Y = ({}^tZ | 0)$ as $Y \mapsto ({}^t[d\rho_1(A)Z + Z\,{}^t d\rho_2(A)] | -{}^tZB)$. Since rank $Z = m_2$, we have $-{}^tZF = (I_{m_2} | 0)$ for some $F \in GL(m_1)$. Then, for any $W \in M(m_2, n - m_1)$, we have $-{}^tZB = W$ with

$$B = F\left(\frac{W}{0} \right).$$

This implies that $d\sigma_n(\mathfrak{G}_n)(x) = W_n$, i.e., x is a generic point of (G_n, σ_n, W_n). Q.E.D.

COROLLARY OF THEOREM 1.16. *Assume that $n \gneq \max\{m_1, m_2\}$. Then a triplet*

$$\left(G \times SL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, V(m_1) \otimes V(n) + V(m_2) \otimes V(n)^*\right)$$

is P.V.-equivalent to the triplet $(G, \rho_1 \otimes \rho_2, V(m_1) \otimes V(m_2))$.

PROOF. It is enough to show that the prehomogeneity of $(G \times GL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*)$ implies that of $(G \times SL(n),\ \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*)$. Any element B of $GL(n)$ can be written as

$$B = B_0 \begin{pmatrix} I_{n-1} & 0 \\ 0 & \det B \end{pmatrix} \quad \text{with } B_0 \in SL(n).$$

Then we have

$$\left(\rho_1(A)(I_{m_1} | 0)\,{}^tB,\ \rho_2(A)({}^tZ | 0)B^{-1}\right) = \left(\rho_1(A)(I_{m_1} | 0)\,{}^tB_0, \rho_2(A)({}^tZ | 0)B_0^{-1}\right)$$

for any $g = (A, B) \in G \times GL(n)$ where $x = ((I_{m_1} | 0), ({}^tZ | 0))$ is a generic point of (G_n, σ_n, W_n) in Theorem 1.16, and hence we obtain our result. Q.E.D.

Now assume that $(G, \rho_1 \otimes \rho_2, V(m_1) \otimes V(m_2))$ is a P.V. We shall study relative invariants of (G_n, σ_n, W_n) for $n \geqslant m_1$ $(\geqslant m_2)$. Let l (resp. l_n) be the number of basic relative invariants of $(G, \rho_1 \otimes \rho_2, V(m_1) \otimes V(m_2))$ (resp. (G_n, σ_n, W_n)).

LEMMA 1.17. (1) *If f is a relative invariant of $(G, \rho_1 \otimes \rho_2, V(m_1) \otimes V(m_2))$, then $F(X, Y) = f(X^tY)$ for $(X, Y) \in W_n = M(m_1, n) \oplus M(m_2, n)$ is a relative invariant of (G_n, σ_n, W_n).*

(2) $l \leqslant l_n \leqslant l + 1$.

PROOF. (1) Since $\sigma_n(g, h)(X, Y) = (\rho_1(g)X^th,\ \rho_2(g)Yh^{-1})$ for $(X, Y) \in W_n$, we have $X^tY \mapsto \rho_1(g)(X^tY)^t\rho_2(g)$ for any $(g, h) \in G \times GL(n)$. Hence we obtain our assertion.

(2) The map $f \mapsto F$ in (1) is injective since $F(X_0, Y_0) = f(Z)$ for any $Z \in M(m_1, m_2) = V(m_1) \otimes V(m_2)$ with $X_0 = (I_{m_1} | 0)$ and $Y_0 = ({}^tZ | 0)$. Hence we have $l \leqslant l_n$. Since $[GL(n), GL(n)] = SL(n)$, we have $(G_n)_1 \supset (G)_1 \times SL(n)$ and hence $G_n/(G_n)_1 \subset G/(G)_1 \times GL(1)$. Since l (resp. l_n) is the rank of the character group of $G/(G)_1$ (resp. $G_n/(G_n)_1$), we have $l_n \leqslant l + 1$. Q.E.D.

PROPOSITION 1.18. (1) *If $n \gneqq m_1$ $(\geqslant m_2)$, we have $l_n = l$, namely, all relative invariants of (G_n, σ_n, W_n) are obtained in the way of (1) in Lemma 1.17.*

(2) *If $n = m_1$ $(\geqslant m_2)$, we have $l_n = l + 1$, and $F(X, Y) = \det X$ for $(X, Y) \in W_n$ is also a relative invariant of (G_n, σ_n, W_n).*

PROOF. (1) It is clear that the generic isotropy subgroup of G_n at the point $x = ((I_{m_1} | 0), ({}^tZ | 0)) \in W_n$ (see Theorem 1.16) contains

$$\left\{ \left(\{1\}, \left(\begin{array}{c|c} I_{m_1} & 0 \\ \hline 0 & A \end{array} \right) \right);\ A \in GL(n - m_1) \right\} = GL(n - m_1).$$

Hence, if $n \gneqq m_1$, then

$$(G_n)_1 = (G_n)_x \cdot \{[G, G] \times SL(n)\} = (G)_1 \times GL(n)$$

and hence $G_n/(G_n)_1 = G/(G)_1$. This implies $l_n = l$.

(2) It is clear that $F(X, Y) = \det X$ is one of the basic relative invariants, which is not of the form $f(X^tY)$. Hence we have $l \lneqq l_n$. By (2) of Lemma 1.17, we have $l_n = l + 1$. Q.E.D.

Now we shall investigate the regularity of (G_n, σ_n, W_n). We denote the Zariski-dense orbit in $V(m_1) \otimes V(m_2) = M(m_1, m_2)$ (resp. W_n) by $M(m_1, m_2)'$ (resp. W_n'). We identify $M(m_1, m_2)$ (resp. W_n) with its dual in the natural way.

Let f be a relative invariant of $(G, \rho_1 \otimes \rho_2,\ M(m_1, m_2))$ and put $\phi = \operatorname{grad} \log f$: $M(m_1, m_2)' \to M(m_1, m_2)$. Since we have

$$\frac{\partial}{\partial x_{ij}} \cdot f(X^tY) = \sum_{k=1}^{m_2} \left(\frac{\partial f}{\partial z_{ik}} \right)(X^tY) \cdot y_{kj}$$

and

$$\frac{\partial}{\partial y_{tj}} \cdot f(X^tY) = \sum_{s=1}^{m_1} \left(\frac{\partial f}{\partial z_{st}} \right)(X^tY) \cdot x_{sj}$$

$(i = 1, \ldots, m_1;\ t = 1, \ldots, m_2;\ j = 1, \ldots, n)$, we get

$$\Phi(X, Y) = \operatorname{grad} \log F(X, Y) = \big(\phi(X^tY)Y,\ {}^t\phi(X^tY)X\big) \tag{1.1}$$

for the relative invariant $F(X, Y) = f(X^tY)$ on W_n where $(X, Y) \in W_n'$.

LEMMA 1.19. *Let (G, ρ, V) be a P.V. with the singular set S. For any $\alpha \in K^{\times}$, we have $\alpha x \in V - S$ for any $x \in V - S$.*

PROOF. Since $d\rho(A)x = 0$ if and only if $d\rho(A)\alpha x = 0$ for $A \in \mathfrak{G} = \mathrm{Lie}(G)$, we have $\mathfrak{G}_{\alpha x} = \mathfrak{G}_x = \{A \in \mathfrak{G};\ d\rho(A)x = 0\}$ and hence αx is a generic point when x is a generic point. Q.E.D.

PROPOSITION 1.20. *Assume that $n \geqslant m_1$ $(\geqslant m_2)$.*

(1) *If $m_1 = m_2$, then $(G, \rho_1 \otimes \rho_2, V(m_1) \otimes V(m_2))$ and (G_n, σ_n, W_n) are regular P.V.'s.*

(2) *If $n = m_1$ and $(G, \rho_1 \otimes \rho_2, V(m_1) \otimes V(m_2))$ is a regular P.V., then (G_n, σ_n, W_n) is also a regular P.V.*

PROOF. (1) Put $m = m_1 = m_2$. Since $(GL(m) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ M(m))$ is a regular P.V., a P.V. $(G, \rho_1 \otimes \rho_2, M(m))$ is also regular by Lemma 1.2. Similarly, if we show that a triplet

$$\big(GL(m) \times GL(m) \times GL(n),\quad \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*,\ M(m,n) \oplus M(m,n)\big)$$

is a regular P.V., we get the regularity of (G_n, σ_n, W_n). However, the isotropy subalgebra at $((I_m\,|\,0), (I_m\,|\,0))$ (cf. Theorem 1.16) is

$$\left\{\left(A, -{}^tA, \left(\begin{array}{c|c} -{}^tA & 0 \\ \hline 0 & B \end{array}\right)\right);\ A \in \mathfrak{gl}(m),\ B \in \mathfrak{gl}(n-m)\right\} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n-m)$$

which is reductive, and hence our P.V. is regular.

(2) For a natural number m, put

$$\Phi_m(X,Y) = \mathrm{grad}\log\{(\det X)^m \cdot F(X,Y)\} = \Phi(X,Y) + (m\,{}^tX^{-1}, 0)$$

(see Lemma 1.21). By (1.1), for a generic point $(X_0, Y_0) \in M(n) \oplus M(m_2, m_1)$ $(n = m_1)$, we have

$$\Phi_m\big(\rho_1(g)X_0{}^th, \rho_2(g)Y_0h^{-1}\big) = \big({}^t\rho_1^{-1}(g)W_0h'^{-1}, {}^t\rho_2^{-1}(g){}^t\phi(Z_0){}^th'\big)$$

where $(g,h) \in G_n = G \times GL(n)$, $Z_0 = X_0{}^tY_0 \in M(m_1, m_2)$, $h' = h{}^tX_0 \in GL(n)$, and $W_0 = mI_n + \phi(Z_0){}^tZ_0 \in M(n)$. If m is sufficiently large, we have $\det W_0 \neq 0$ and for any given $X \in GL(n)$, there exist $g \in G$ and $h' \in GL(n)$ (hence $h \in GL(n)$) such that ${}^t\rho_1^{-1}(g)W_0h'^{-1} = X$, i.e., $h' = X^{-1}\,{}^t\rho_1^{-1}(g)W_0$. Hence Φ_m is dominant if and only if $\{{}^t\rho_1^{-1}(g)W_0\phi(Z_0)\rho_2^{-1}(g);\ g \in G\}$ is Zariski-dense in $M(m_1, m_2)$, namely, $W_0\phi(Z_0)$ is a generic point of $(G, (\rho_1 \otimes \rho_2)^*, M(m_1, m_2))$. By assumption, $\phi(Z_0)$ is a generic point and hence $\phi(Z_0) + (1/m)\phi(Z_0){}^tZ_0\phi(Z_0)$ is also a generic point for a sufficiently large m. By Lemma 1.19, $W_0\phi(Z_0)$ is a generic point for such m. This implies Φ_m is dominant for sufficiently large m and (G_n, σ_n, W_n) is regular. Q.E.D.

LEMMA 1.21. *Let f be a relative invariant of a P.V. $(H \times GL(m_2),\ \rho \otimes \Lambda_1,\ V(m_1) \otimes V(m_2))$ with $m_1 \geqslant m_2 \geqslant 1$. By $\langle X, Y\rangle = \mathrm{Tr}\,{}^tXY$ for $X, Y \in M(m_1, m_2) = V(m_1) \otimes V(m_2)$, we identify $M(m_1, m_2)$ with its dual. Put $\phi(Z) = \mathrm{grad}\log f(Z) \in M(m_1, m_2)$. Then we have ${}^tZ\phi(Z) = rI_{m_2}$, with $r = (\deg f)/m_2$ for any generic point Z in $M(m_1, m_2)$. Hence $\mathrm{rank}\,\phi(Z) = m_2$. In particular, when $m_1 = m_2$ and $f(Z) = \det Z$, we have $\phi(Z) = {}^tZ^{-1}$.*

PROOF. Since f is a relative invariant, there exists a character χ of H and an integer r satisfying $f(\rho(h)Z{}^tB) = \chi(h)\cdot(\det B)^r \cdot f(Z)$ for $g = (h, B) \in H \times GL(m_2)$ and a generic point Z in $M(m_1, m_2)$. Then we have $r = (\deg f)/m_2$ since $f(Z{}^tB) = s^{\deg f} f(Z)$ and $(\det B)^r = s^{rm_2}$ for $g = (1, sI_{m_2})$. By (2) of Proposition 9, p. 62 in [**1**], we have $\operatorname{Tr} A({}^tZ\phi(Z)) = \langle Z{}^tA, \phi(Z)\rangle = r \operatorname{Tr} A$ for all $A \in \mathfrak{gl}(m_2)$ and hence ${}^tZ\phi(Z) = rI_{m_2}$ with $r = (\deg f)/m_2$. Q.E.D.

PROPOSITION 1.22. (1) *If $n \gneq m_1 \gneq m_2$, then (G_n, σ_n, W_n) is a nonregular P.V.*

(2) *Assume that $n = m_1 \gneq m_2$ and $(G, \rho_1 \otimes \rho_2, M(m_1, m_2))$ is a nonregular P.V. which satisfies at least one of the following conditions* (i)–(iv). *Then (G_n, σ_n, W_n) is a nonregular P.V.*

(i) *G = reductive.*

(ii) *$G = H \times GL(m_2)$, $\rho_1\colon H \to GL(m_1)$ and $\rho_2 = \Lambda_1$ where H is not reductive.*

(iii) rank $\phi(Z) \lneq m_2$ *for any relative invariant f where* $\phi(Z) = \operatorname{grad} \log f(Z) \in M(m_1, m_2)$.

(iv) *The dual $(G, (\rho_1 \otimes \rho_2)^*, M(m_1, m_2))$ is a non-P.V.*

PROOF. (1) Since rank $\phi(X{}^tY) \leqslant m_2 \lneq m_1$ in (1.1), $\Phi(X, Y)$ cannot be dominant. Hence (G_n, σ_n, W_n) is not regular by (1) of Proposition 1.18.

(2) (i) The generic isotropy subgroups of $(G, \rho_1 \otimes \rho_2, M(m_1, m_2))$ and (G_n, σ_n, W_n) are isomorphic. Hence their reductivity is equivalent to the regularity when G is reductive.

(ii) We use the same notation as in the proof of (2) in Proposition 1.20. The P.V. (G_n, σ_n, W_n) is regular if and only if Φ_m is dominant for some m, and it is so if and only if $W_0\phi(Z_0) = m\phi(Z_0) + \phi(Z_0){}^tZ_0\phi(Z_0)$ is a generic point for some m, where $\phi(Z_0)$ is a nongeneric point. By Lemma 1.21, we have $W_0\phi(Z_0) = (m + (\deg f)/m_2)\phi(Z_0)$, which is not a generic point for any m. Note that we may assume that the dual of $(G, \rho_1 \otimes \rho_2)$ is a P.V. since otherwise we deal in the case (iv). Hence (G_n, σ_n, W_n) is not regular.

(iii) Since $\Phi_m(X, Y) = (\phi(X{}^tY)Y + m{}^tX^{-1}, {}^t\phi(X{}^tY)X)$ and rank ${}^t\phi(X{}^tY) \lneq m_2$, the map Φ_m cannot be dominant.

(iv) If (G_n, σ_n, W_n) is a regular P.V., then its dual is also a regular P.V. However, the dual is a P.V. if and only if$(G,(\rho_1 \otimes \rho_2)^*)$ is a P.V. and hence we obtain our assertion. Q.E.D.

EXAMPLE 1.23. Consider the following P.V.'s, $(G, \rho_1 \otimes \rho_2, M(m_1, m_2))$.

(A) $(\mathrm{Sp}(n) \times GO(3), \Lambda_1 \otimes \Lambda_1, M(2n, 3))$.

(B) $(\mathrm{Tr}_u(m_1) \times GL(m_2), \Lambda_1 \otimes \Lambda_1, M(m_1, m_2))$ where $\mathrm{Tr}_u(m_1)$ denotes the upper triangular matrix group of degree m_1 ($\gneq m_2$).

(C) $(\{\left(\begin{smallmatrix}1 & b\\ 0 & a\end{smallmatrix}\right);\ a \neq 0\}, \Lambda_1, M(2,1))$

Then (A) (resp. (B); (C)) satisfies the conditions (i) and (iii) (resp. (ii); (ii), (iii) and (iv)) in Proposition 1.22.

REMARK 1.24. Let $(G, \rho_1 \otimes \rho_2, M(m_1, m_2))$ be a P.V. with a relative invariant f. Put

$$\phi(Z) = \left(\frac{1}{f(Z)}\frac{\partial f}{\partial z_{ij}}(Z)\right) \quad \text{for } Z = (z_{ij}) \in M(m_1, m_2).$$

In the case of Lemma 1.21, $\phi(Z)^t Z\phi(Z)$ coincides with $\phi(Z)$ up to a constant multiple. However, this is not true in general. Counterexample: $m_1 = m+1$, $m_2 = m \geqslant 2$, $G = GL(m) \times GL(1) \times SO(m)$, $\rho_1 = \Lambda_1 \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes 1$, $\rho_2 = 1 \otimes 1 \otimes \Lambda_1$, $f(Z) = (\det X)(y_1^2 + \cdots + y_m^2)$ for $Z = \left[\begin{smallmatrix} X \\ y \end{smallmatrix}\right] \in M(m+1, m)$ ($X \in M(m)$, $y \in V(m)$).

Now let us consider a triplet (G_n, σ_n, W_n) when $m_1 \gneq n$. For an element X of $M(m_1, n)$, we denote by $\langle X \rangle$ the vector subspace of $V(m_1)$ spanned by column vectors of X. If rank $X = n$, then $\langle X \rangle$ is an element of the Grassmann variety $\mathrm{Grass}_n(V(m_1))$.

Theorem 1.25. *Assume that $m_1 \gneq n$. Then the following conditions are equivalent.*

(1) *(G_n, σ_n, W_n) is a P.V.*

(2) *A variety*

$$U \overset{\text{def}}{=} \{(\langle X \rangle, X^t Y) \in \mathrm{Grass}_n(V(m_1)) \oplus M(m_1, m_2);$$

$$(X, Y) \in W_n = M(m_1, n) \oplus M(m_2, n), \operatorname{rank} X = n, \operatorname{rank} Y = \min\{n, m_2\}\}$$

is G-prehomogeneous, i.e., U has a Zariski-dense G-orbit.

(3) (i) *$(G \times GL(n), \rho_1 \otimes \Lambda_1, M(m_1, n))$ is a P.V., and* (ii) *$(G_{\langle W \rangle}, \tilde{\rho}_1 \otimes (\rho_2 | G_{\langle W \rangle}), \langle W \rangle \otimes V(m_2))$ is a P.V. where $W \in M(m_1, n)$ is a generic point of* (i) *and $G_{\langle W \rangle}$ is the isotropy subgroup of G at $\langle W \rangle \in \mathrm{Grass}_n(V(m_1))$. Here $\tilde{\rho}_1$ is the representation of $G_{\langle W \rangle}$ on $\langle W \rangle$ induced by ρ_1.*

Proof. Put

$$U' = \{(X, Y) \in W_n = M(m_1, n) \oplus M(m_2, n);$$
$$\operatorname{rank} X = n, \operatorname{rank} Y = \min\{n, m_2\}\}.$$

Then (G_n, σ_n, W_n) is a P.V. if and only if U' is a $G \times GL(n)$-prehomogeneous variety. Define a map φ: $U' \to U$ by $\varphi(X, Y) = (\langle X \rangle, X^t Y)$ for $(X, Y) \in U'$. Clearly it is surjective and $G \times GL(n)$-equivariant where $GL(n)$ acts on U trivially. In view of Lemma 5, p. 36 in [**1**], to see the equivalence of (1) and (2), it is enough to show that each fiber of φ is $GL(n)$-homogeneous. Assume that $(\langle X \rangle, X^t Y) = (\langle X' \rangle, X'^t Y')$. Since $\langle X \rangle = \langle X' \rangle$, we have $X' = X^t B$ for some $B \in GL(n)$, and hence $0 = X'^t Y' - X^t Y = X({}^t B^t Y' - {}^t Y)$. Since rank $X = n$, we have $Y' = YB^{-1}$, i.e., $(X', Y') = (X^t B, YB^{-1})$ for some $B \in GL(n)$. Now define a map ψ: $U \to \mathrm{Grass}_n(V(m_1))$ by $\psi(\langle X \rangle, X^t Y) = \langle X \rangle$. Clearly it is surjective and G-equivariant. By p. 37 in [**1**], the Grassmann variety $\mathrm{Grass}_n(V(m_1))$ is G-prehomogeneous if and only if (i) holds. Since the fiber

$$\psi^{-1}(\langle W \rangle) \cong \{X^t Y \in M(m_1, m_2) = V(m_1) \otimes V(m_2);$$
$$\operatorname{rank} X = n, \operatorname{rank} Y = \min\{n, m_2\}, \langle X \rangle = \langle W \rangle\}$$

is isomorphic to $\langle W \rangle \otimes V(m_2)$, we have the equivalency of (2) and (3). Q.E.D.

Corollary 1.26. *Assume that $m_1 \geqslant m_2 \gneq n$. Then (G_n, σ_n, W_n) is a P.V. if and only if*

(i) *$(G \times GL(n), \rho_1 \otimes \Lambda_1, M(m_1, n))$ is a P.V.*

(ii) *$(G_{\langle W_1 \rangle} \times GL(n), \rho_2 | G_{\langle W_1 \rangle} \otimes \Lambda_1, M(m_2, n))$ is a P.V.*

(iii) *$(G_{\langle W_1, W_2 \rangle}, \tilde{\rho}_1 \otimes \tilde{\rho}_2, \langle W_1 \rangle \otimes \langle W_2 \rangle)$ is a P.V.*

Here $W_1 \in M(m_1, n)$ (resp. $W_2 \in M(m_2, n)$) is a generic point of (i) (*resp.* (ii)) *and* $G_{\langle W_1, W_2\rangle}$ *is the isotropy subgroup of* G *at* $(\langle W_1\rangle, \langle W_2\rangle) \in \mathrm{Grass}_n(V(m_1)) \oplus \mathrm{Grass}_n(V(m_2))$. *Here* $\tilde\rho_1 \otimes \tilde\rho_2$ *is the representation of* $G_{\langle W_1, W_2\rangle}$ *on* $\langle W_1\rangle \otimes \langle W_2\rangle$ *induced by* $\rho_1 \otimes \rho_2$.

PROOF. By the proof of Theorem 1.25,

$$U'' = \{(\langle X\rangle, \langle Y\rangle, X{}^tY) \in \mathrm{Grass}_n(V(m_1)) \oplus \mathrm{Grass}_n(V(m_2)) \oplus M(m_1, m_2);\ \operatorname{rank} X = \operatorname{rank} Y = n\}$$

is isomorphic to $U = \{(\langle X\rangle, X{}^tY)\}$. Define a map $f\colon U'' \to \mathrm{Grass}_n(V(m_1)) \oplus \mathrm{Grass}_n(V(m_2))$ by $f(\langle X\rangle, \langle Y\rangle, X{}^tY) = (\langle X\rangle, \langle Y\rangle)$. Clearly it is surjective and G-equivariant. Note that $\mathrm{Grass}_n(V(m_1)) \oplus \mathrm{Grass}_n(V(m_2))$ is G-prehomogeneous if and only if (i) and (ii) hold. Since a fiber $f^{-1}(\langle W_1\rangle, \langle W_2\rangle) \cong \langle W_1\rangle \otimes \langle W_2\rangle$, we have our result by Lemma 5, p. 36 in [**1**]. Q.E.D.

Finally we shall prove the invariance of regularity under a castling transformation without assuming the reductivity of a group. It is enough to prove that if a triplet $(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ $(m \gneq n \geqslant 1)$ is a regular P.V., then a castling transform $(G \times GL(m-n),\ \rho^* \otimes \Lambda_1,\ V(m)^* \otimes V(m-n))$ is also a regular P.V. Here we do not assume that G is reductive, but we may assume that $\rho(G) \subset SL(m)$ and $\rho^*(G) \subset SL(m)$ without loss of generality. By $\langle X, Y\rangle = \operatorname{Tr} X{}^tY$ for $X, Y \in M(m, n)$, we identify $V(m) \otimes V(n) = M(m, n)$ with its dual. Take a generic point X_0 in $M(m, n) = V(m) \otimes V(n)$. Let f be a nondegenerate relative invariant and χ its corresponding character. Put $Y_0 = \operatorname{grad} \log f(X_0)$. Then, by Lemma 1.21, we have ${}^tX_0Y_0 = ((\deg f)/n)I_n$. Put $Y_0' = (n/(\deg f))Y_0$ and $P = X_0{}^tY_0' \in M(m)$.

LEMMA 1.27. (1) $P^2 = P$, (2) $\operatorname{Tr} P = n$, (3) $\operatorname{Tr} d\rho(A)P = (n/(\deg f))\delta\chi(A)$ *for all* $A \in \mathfrak{G} = \mathrm{Lie}(G)$.

PROOF. (1) and (2) are obvious. By (2) of Proposition 9, p. 62 in [**1**], we have

$$\begin{aligned}\operatorname{Tr} d\rho(A)P &= \langle d\rho(A)X_0, Y_0'\rangle \\ &= \frac{n}{\deg f}\langle d\rho(A)X_0, \operatorname{grad}\log f(X_0)\rangle = \frac{n}{\deg f}\delta\chi(A) \quad \text{for } A \in \mathfrak{G}. \quad \text{Q.E.D.}\end{aligned}$$

Since $PX_0 = X_0{}^tY_0'X_0 = X_0$ and $\dim P(V(m)) \leqslant \dim {}^tY_0'(V(m)) \leqslant n$, we have $P(V(m)) = \langle X_0\rangle$ and it is a generic point of $(G, \mathrm{Grass}_n(V(m)))$.

Now we define the adjoint P^* of P by $\langle PX, Y\rangle = \langle X, P^*Y\rangle$ for $X, Y \in M(m, n)$. Clearly we have $P^* = {}^tP \in M(m)$. Similarly, since $P^*Y_0' = {}^tPY_0' = Y_0'{}^tX_0Y_0' = Y_0'$ and Y_0' is a generic point of $(G \times GL(n),\ \rho^* \otimes \Lambda_1^*,\ M(m, n))$, $P^*(V^*(m)) = \langle Y_0'\rangle$ is a generic point of $(G, \mathrm{Grass}_n(V(m)^*))$.

LEMMA 1.28. (1) $(I_m - P)(V(m)) = P^*(V^*(m))^\perp$,
(2) $(I_m - P^*)(V^*(m)) = P(V(m))^\perp$.

PROOF. For any $x \in V(m)$, we have ${}^tY_0'Px = {}^tY_0'X_0{}^tY_0'x = {}^tY_0'x$, i.e., ${}^tY_0'(I_m - P)x = 0$. Since $P^*(V^*(m)) = \langle Y_0'\rangle$, $P^*(V^*(m))$ annihilates $(I_m - P)(V(m))$ and hence $(I_m - P)(V(m)) \subset P^*(V^*(m))^\perp$. Since both sides have dimension $m - n$, they coincide. (2) is similarly obtained as (1). Q.E.D.

LEMMA 1.29. *Put* $Q^* = I_m - P^*$ *and* $Q = (Q^*)^*$. *Then we have* (1) $Q^{*2} = Q^*$, (2) $\operatorname{Tr} Q^* = m - n$, (3) $\operatorname{Tr} d\rho^*(A)Q^* = (n/(\deg f))\delta\chi(A)$ *for* $A \in \mathfrak{G}$, (4) $Q^*(V^*(m))$ *is a generic point of* $(G, \operatorname{Grass}_{m-n}(V^*(m)))$, (5) $Q(V(m))$ *is a generic point of* $(G, \operatorname{Grass}_{m-n}(V(m)))$.

PROOF. (1) and (2) are obvious. Since $\rho^*(G) \subset SL(m)$, we have $\operatorname{Tr} d\rho^*(A) = 0$ for $A \in \mathfrak{G}$, and hence

$$\begin{aligned}\operatorname{Tr} d\rho^*(A)Q^* &= -\operatorname{Tr} d\rho^*(A)Y_0'{}^tX_0 = -\langle X_0, d\rho^*(A)Y_0'\rangle \\ &= \langle d\rho(A)X_0, Y_0'\rangle = \frac{n}{\deg f}\delta\chi(A) \quad \text{for } A \in \mathfrak{G},\end{aligned}$$

i.e., (3). For (4), by (2) of Lemma 1.28, $Q^*(V^*(m)) = P(V(m))^{\perp} = \langle X_0\rangle^{\perp}$ and hence it is a generic point of $(G, \operatorname{Grass}_{m-n}(V^*(m)))$. Similarly we obtain (5). Q.E.D.

Since $\dim Q^*(V^*(m)) = m - n$, by taking basis, we can express $Q^*\colon V^*(m) \to Q^*(V^*(m))$ (resp. $Q^*(V^*(m)) \to V^*(m)$) by ${}^tZ_0$ (resp. W_0) for some Z_0, $W_0 \in M(m, m-n)$. Then we have $Q^* = W_0{}^tZ_0$ and ${}^tZ_0W_0 = I_{m-n}$. Since $Q^*W_0 = W_0$ and $\operatorname{rank} W_0 = m - n$ by ${}^tZ_0W_0 = I_{m-n}$, we have $Q^*(V^*(m)) = \langle W_0\rangle$. Hence, by (4) of Lemma 1.29, $\langle W_0\rangle$ is a generic point of $(G, \operatorname{Grass}_{m-n}(V^*(m)))$. This implies that $W_0 \in M(m, m-n) = V(m)^* \otimes V(m-n)$ is a generic point of

$$\big(G \times GL(m-n),\ \rho^* \otimes \Lambda_1,\ V(m)^* \otimes V(m-n)\big).$$

Since $Q = {}^tQ^* = Z_0{}^tW_0$ and $QZ_0 = Z_0$, we have $Q(V(m)) = \langle Z_0\rangle$ and hence Z_0 is a generic point of $(G \times GL(m-n), \rho \otimes \Lambda_1^*, V(m) \otimes V^*(m-n))$. Now we have

$$\begin{aligned}\langle d\rho^*(A)W_0 + W_0{}^tB, Z_0\rangle &= \operatorname{Tr} d\rho^*(A)W_0{}^tZ_0 + \operatorname{Tr}{}^tB{}^tZ_0W_0 \\ &= \operatorname{Tr} d\rho^*(A)Q^* + \operatorname{Tr}{}^tB \\ &= \frac{n}{\deg f}\delta\chi(A) + \operatorname{Tr} B \quad \text{for all } (A, B) \in \mathfrak{G} \oplus \mathfrak{gl}(m-n).\end{aligned}$$

Let F be a relative invariant of $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V^*(m) \otimes V(m-n))$ corresponding to f (see Proposition 18, p. 68 in **[1]**). Then its infinitesimal character is given by

$$\delta\chi(A) + \frac{\deg F}{(m-n)}\operatorname{Tr} B \quad \text{for } (A, B) \in \mathfrak{G} \oplus \mathfrak{gl}(m-n)$$

(cf. the proof of Lemma 1.21). Since $(\deg F)/(m-n) = (\deg f)/n$ by p. 68 in **[1]**, we have

$$\left\langle d\rho^*(A)W_0 + W_0{}^tB, \frac{\deg f}{n}Z_0 - \operatorname{grad}\log F(W_0)\right\rangle = 0$$

for all $(A, B) \in \mathfrak{G} \oplus \mathfrak{gl}(m-n)$. Since W_0 is a generic point, we have

$$\{d\rho^*(A)W_0 + W_0{}^tB;\ (A, B) \in \mathfrak{G} \oplus \mathfrak{gl}(m-n)\} = V(m)^* \otimes V(m-n).$$

Hence $\operatorname{grad}\log F(W_0) = ((\deg f)/n)Z_0$, which is a generic point. This implies that the map $\operatorname{grad}\log F$ is dominant and $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V(m)^* \otimes V(m-n))$ is a regular P.V. Thus we obtain the following theorem.

THEOREM 1.30 (M. SATO AND T. KIMURA). *The regularity of P.V.'s is a property invariant under a castling transformation.*

DEFINITION 1.31. Assume that (G, ρ_1, V_1) is a P.V. with a generic isotropy subgroup H. Then a triplet $(G, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$ is a P.V. if and only if $(H, \rho_2|_H, V_2)$ is a P.V. by Lemma 5, p. 36 in [1]. Sometimes we say that $(G, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$ and $(H, \rho_2|_H, V_2)$ are *isotropic P.V.-equivalent* to each other.

REMARK 1.32. Let (G, ρ_1, V_1) be a P.V. with a reductive generic isotropy subgroup H. Then a triplet $(G, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$ is a P.V. if and only if $(G, \rho_1 \oplus \rho_2^*, V_1 \oplus V_2^*)$ is a P.V. We say that they are *reductive P.V.-equivalent*. Their generic isotropy subgroups are isomorphic to each other.

PROPOSITION 1.33 (O. YASUKURA). *$(GL(2m+1) \times H,\ \Lambda_2 \otimes 1 + \rho \otimes \rho'$ (resp. $\Lambda_2^* \otimes 1 + \rho \otimes \rho'$)) is a P.V. if and only if $(Sp(m) \times GL(2m+1) \times H,\ \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \rho \otimes \rho'$ (resp. $\Lambda_1 \otimes \Lambda_1^* \otimes 1 + 1 \otimes \rho \otimes \rho'$)) is a P.V.*

PROOF. Since the $GL(2m+1)$-part of generic isotropy subgroups of $(Sp(m) \times GL(2m+1),\ \Lambda_1 \otimes \Lambda_1$ (resp. $\Lambda_1 \otimes \Lambda_1^*$)) and $(GL(2m+1), \Lambda_2$ (resp. Λ_2^*)) coincide, we get our result. Q.E.D.

PROPOSITION 1.34. (1) *If $(G \times GL(2m+1), \rho \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + \sigma \otimes 1)$ is a P.V., then $(G \times Sp(m), \rho \otimes \Lambda_1 + \sigma \otimes 1)$ is also a P.V. Moreover, they are P.V.-equivalent when* $\deg \rho \leqslant 2m+1$.

(2) *Assume that* $\deg \rho \leqslant 2m+1$. *Then $(G \times GL(2m+1),\ \rho \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1)$ is P.V.-equivalent to $(G \times GL(\deg \rho - 1),\ \rho^* \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1)$.*

PROOF. (1) is an immediate consequence of Proposition 1.33, Theorems 1.14 and 1.16. For (2), by Proposition 1.33, it is P.V.-equivalent to

$$(G \times Sp(m) \times GL(2m+1), (\rho \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1 + \sigma \otimes 1 \otimes 1),$$

which is castling equivalent to

$$\left(G \times Sp(m) \times GL(\deg \rho - 1), \left(\rho^* \otimes 1 + 1 \otimes \Lambda_1^*\right) \otimes \Lambda_1 + \sigma \otimes 1 \otimes 1\right).$$

By Proposition 13, p. 40 in [1], we have our result. Q.E.D.

PROPOSITION 1.35. *Assume that*

$$\left(GL(1)^{2+s+t} \times G \times SL(n),\ \rho \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \left(\Lambda_1^* + \tau_1 + \cdots + \tau_t\right)\right) \qquad (m = \deg \rho \leqslant n)$$

is a P.V. Then

$$\left(GL(1)^{2+s+t} \times G \times SL(n),\ \rho \otimes \Lambda_1 + (\rho + \sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)\right)$$

must be a P.V.

PROOF. Let $x = ((I_m|0), y, z)$ be a generic point of $\rho \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$. Then the $G \times SL(n)$-part of the isotropy subalgebra $\mathfrak{h}$ is given by

$$\left\{(A, B) \mid B = \left(\begin{array}{c|c} -{}^t d\rho(A) & C \\ \hline 0 & D \end{array}\right), d(\sigma_1 + \cdots + \sigma_s)(A)y + \mathfrak{gl}(1)^s \cdot y = 0, \right.$$
$$\left. d(\tau_1 + \cdots + \tau_t)(B)z + \mathfrak{gl}(1)^t \cdot z = 0\right\}.$$

Since $\mathfrak{gl}(1) + \Lambda^*(B)$ induces

$$\left(\begin{array}{c|c} d\rho(A) & 0 \\ \hline -{}^tC & -{}^tD \end{array}\right)\left(\frac{X_1}{X_2}\right) + \alpha\left(\frac{X_1}{X_2}\right) = \left(\frac{\alpha X_1 + d\rho(A)X_1}{*}\right),$$

we have our result. Q.E.D.

PROPOSITION 1.36. *Assume that* $\deg \rho = 2n + 1 \lneq 2m + 1$. *Then*

$$\big(GL(1)^{3+s} \times G \times SL(2m + 1),$$
$$\rho \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_2 + 1 \otimes \Lambda_1^*\big)$$

is P.V.-equivalent to

$$\big(GL(1)^{2+s} \times G,\ \Lambda^2(\rho)^* + \rho + \sigma_1 + \cdots + \sigma_s\big).$$

PROOF. Let

$$\left((I_{2n+1}0), \left(\begin{array}{c|c} X & Y \\ \hline -{}^tY & Z \end{array}\right)\right)$$

be a generic point of $\rho \otimes \Lambda_1 + 1 \otimes \Lambda_2$. Then we have $\det Z \neq 0$. By the action of

$$\{1\} \times \left(\begin{array}{c|c} I_{2n+1} & -YZ^{-1} \\ \hline 0 & I_{2m-2n} \end{array}\right) \in G \times SL(2m + 1),$$

we may assume that $Y = 0$. Then the $G \times GL(2m + 1)$-part of the isotropy subgroup at this point is given by

$$\left\{\left(A, \left(\begin{array}{c|c} {}^t\rho^{-1}(A) & 0 \\ \hline 0 & B \end{array}\right)\right); A \in G, (GL(1) \times)\Lambda_2^*(\rho)(A)X = X, B \in Sp(m - n)\right\}.$$

Since

$$\Lambda_1^*\left(\left(\begin{array}{c|c} {}^t\rho^{-1}(A) & 0 \\ \hline 0 & B \end{array}\right)\right) = \left(\begin{array}{c|c} \rho(A) & 0 \\ \hline 0 & {}^tB^{-1} \end{array}\right)$$

and $(Sp(m - n), \Lambda_1^*)$ is a P.V., we have our result. Q.E.D.

PROPOSITION 1.37. *If G is reductive and* $\deg \rho = 2n' < 2m + 1$, *then*

$$\big(GL(1)^{2+s} \times G \times SL(2m + 1), \rho \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\sigma_1 + \cdots + \sigma_s) \otimes 1\big)$$

is P.V.-equivalent to

$$\big(GL(1)^{2+s} \times G \times SL(2m + 1), \rho \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\sigma_1 + \cdots + \sigma_s)^* \otimes 1\big).$$

PROOF. By (2) of Proposition 1.34, it is P.V.-equivalent to

$$\left(GL(1)^{2+s} \times G \times SL(2n'-1),\ \rho \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\sigma_1 + \cdots + \sigma_s)^* \otimes 1\right).$$

By Proposition 1.33, it is P.V.-equivalent to

$$\left(GL(1)^{2+s} \times G \times Sp(n'-1) \times SL(2n'-1),\right.$$
$$\left.(\sigma_1 + \cdots + \sigma_s)^* \otimes 1 \otimes 1 + (\rho \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1\right).$$

Since $2n' + 2(n'-1) - (2n'-1) = 2n'-1$, it is castling-equivalent to

$$\left(GL(1)^{2+s} \times G \times Sp(n'-1) \times SL(2n'-1),\right.$$
$$\left.(\sigma_1 + \cdots + \sigma_s) \otimes 1 \otimes 1 + (\rho \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1\right).$$

Hence we obtain our result. Q.E.D.

REMARK 1.38. In 1971, Mikio Sato gave the P.V.-equivalence of $(G, \rho_1 \otimes \rho_2, V(m_1) \otimes V(m_2))$ and (G_n, σ_n, W_n) for $n \geqslant \max\{m_1, m_2\}$ in his lecture held at the University of Tokyo (Theorems 1.14, 1.16). The P.V.-equivalence of (1) and (2) in Theorem 1.25 for $m_1 \gneq n \geqslant m_2$ was obtained by Yasuo Teranishi and Sigehumi Mori independently in 1976. For $m_1 \geqslant m_2 \gneq n$, they also showed that (G_n, σ_n, W_n) is a P.V. if and only if U'' (in the proof of Corollary 1.26) is G-prehomogeneous. In 1985, M. Taguchi pointed out that $U'' \cong U$ and got a result similar to the one for $m_1 \gneq n \geqslant m_2$. In 1985, Masaki Kashiwara pointed out to T. Kimura the equivalence of (2) and (3) in Theorem 1.25, and Corollary 1.26.

2. The case for $(\tau_1, \ldots, \tau_t) \neq (\Lambda_1^{(*)}, \ldots, \Lambda_1^{(*)})$. In the rest of this paper, we shall classify all 2-simple P.V.'s by using the P.V.-equivalences obtained in §1. All 2-simple P.V.'s of type I have already been classified [3]. Therefore, to complete the classification, it is enough to classify all 2-simple P.V.'s of the following type, which is called type II. Namely, we shall consider

$$\left(GL(1)^{k+s+t} \times G_1 \times SL(n),\ \rho_1 \otimes \Lambda_1^{(*)} + \cdots + \rho_k \otimes \Lambda_1^{(*)}\right.$$
$$\left.+ (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)\right)$$

with $2 \leqslant \deg \rho_i \leqslant n$ $(i = 1, \ldots, k)$, where G_1 is a simple algebraic group and $\Lambda_1^{(*)}$ stands for Λ_1 or its dual Λ_1^*. By Proposition 1.15, we may assume that

$$\rho_1 \otimes \Lambda_1^{(*)} + \cdots + \rho_k \otimes \Lambda_1^{(*)} = (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1$$

without loss of generality. Then we may also assume that $2 \leqslant \deg \rho_i \lneq n$ for all $i = 1, \ldots, k$ by Proposition 1.12 and Remark 1.13.

In this section, we shall consider the case when $(\tau_1 + \cdots + \tau_t) \neq (\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$. We may assume that $\tau_1 \neq \Lambda_1^{(*)}$. Then, by [1], τ_1 must be one of $\Lambda_2^{(*)}$, $2\Lambda_1^{(*)}$ or $\Lambda_3^{(*)}$ $(n = 6, 7, 8)$. We may omit the case $\tau_1 = 3\Lambda_1$ with $n = 2$ since $n \gneq \deg \rho_i \geqslant 2$ $(i = 1, \ldots, k)$ by our assumption.

THEOREM 2.1. *If* $\tau_1 = \Lambda_2^{(*)}$ *with* $n = 2m$ (*resp.* $2\Lambda_1^{(*)}$ *with any* n, $\Lambda_3^{(*)}$ *with* $n = 7$), *then*

$$\left(GL(1)^l \times G_1 \times SL(n), (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1\right.$$
$$\left.+ (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)\right)$$

is a 2*-simple P.V. of type* II *if and only if* $\tau_2 + \cdots + \tau_t = \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}$ *and*

$$\big(GL(1)^{l-1} \times H \times G_1,\ \rho \otimes (\rho_1 + \cdots + \rho_k) \\ + \big(\big(\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}\big) \cdot \rho\big) \otimes 1 + 1 \otimes (\sigma_1 + \cdots + \sigma_s)\big)$$

is a 2*-simple P.V. of type* I *where* $(H, \rho) = (Sp(m), \Lambda_1)$ (*resp.* $(SO(n), \Lambda_1)$, $((G_2), \Lambda_2)$) *and* $l = k + s + t$.

PROOF. Since the generic isotropy subgroup of $(GL(2m),\ \Lambda_2^{(*)}, V(2m^2 - m))$ (resp. $(GL(n),\ 2\Lambda_1^{(*)},\ V(n))$, $(GL(7),\ \Lambda_3^{(*)},\ V(35))$ is $Sp(m)$ (resp. $O(n)$, (G_2)) (see p. 76 (resp. p. 75, p. 86) in [1]), we have our result. Q.E.D.

When $n = 8$, we have $\tau_1 \neq \Lambda_3$, namely, we have the following proposition.

PROPOSITION 2.2. *A triplet* $(GL(1)^2 \times G_1 \times SL(8),\ \rho \otimes \Lambda_1 + 1 \otimes \Lambda_3,\ V(d) \otimes V(8) + V(56))$ *with* $d = \deg \rho \leqslant 7$, *is a non-P.V.*

PROOF. We may assume that $(G_1, \rho) = (SL(d), \Lambda_1)$ since $\rho(G_1) \subset SL(d)$. If it is a P.V., then we have $\dim G = d^2 + 64 \geqslant \dim V = 8d + 56$ $(2 \leqslant d \leqslant 7)$ and hence $d = 7$. In this case, a castling transform $(GL(1)^2 \times SL(8),\ \Lambda_1^* + \Lambda_3,\ V(8)^* + V(56))$ of this triplet is not a P.V. by [2]. Thus we have our result. Q.E.D.

Now let us consider the case when $n = 6$ and $\tau_1 = \Lambda_3$.

LEMMA 2.3. *Assume that* $(GL(1)^2 \times G_1 \times SL(6),\ \rho_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3)$ *with* $2 \leqslant \deg \rho_1 \leqslant 5$, *is a P.V. Then* (G_1, ρ_1) *must be one of* $(SL(2), \Lambda_1)$, $(SL(4), \Lambda_1)$, $(Sp(2), \Lambda_1)$ *or* $(SL(5), \Lambda_1)$.

PROOF. Since $2 \leqslant \deg \rho_1 \leqslant 5$ and G_1 is a simple algebraic group, (G_1, ρ_1) must be one of the ones above or $(SL(3), \Lambda_1)$, $(SL(2), 2\Lambda_1) = (SO(3), \Lambda_1)$, $(SL(2), 3\Lambda_1)$, $(SL(2), 4\Lambda_1)$, or $(Sp(2), \Lambda_2) = (SO(5), \Lambda_1)$. By p. 80 in [1],

$$\big(GL(1)^2 \times G_1 \times SL(6),\ \rho_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3\big)$$

and

$$\big(GL(1) \times G_1 \times SL(3) \times SL(3),\ \Lambda_1 \otimes (\rho_1 \otimes \Lambda_1 \otimes 1 + \rho_1 \otimes 1 \otimes \Lambda_1)\big)$$

are P.V.-equivalent. When $\deg \rho_1 = 2$, then the latter is a P.V. of trivial type. When $\deg \rho_1 = 3$, then the latter representation space can be identified with $V = M(3) \oplus M(3)$ and $f(x) = (\det X) \cdot (\det Y)^{-1}$ for $x = (X, Y) \in V$ is a nonconstant absolute invariant, and hence, it cannot be a P.V. When $\deg \rho_1 = 4$, it is P.V.-equivalent to a castling transform $(GL(1) \times G_1,\ \Lambda_1 \otimes (\rho_1^* + \rho_1^*))$ of the latter space. Hence, $(G_1, \rho_1) \neq (SL(2), 3\Lambda_1)$ and $(G_1, \rho_1) = (Sp(2), \Lambda_1)$ or $(SL(4), \Lambda_1)$ when $\deg \rho_1 = 4$. When $\deg \rho_1 = 5$, then $\dim G \geqslant \dim V$ in the latter space implies that $\dim G_1 \geqslant 13$ and hence $(G_1, \rho_1) \neq (SL(2),\ 4\Lambda_1)$, $(SO(5), \Lambda_1)$. When $(G_1, \rho_1) = (SL(5), \Lambda_1)$, a castling transform $(GL(5) \times (SL(2) \times SL(2)),\ \Lambda_1 \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1))$ of the latter space is a trivial P.V. Q.E.D.

THEOREM 2.4. *All nonirreducible* 2*-simple P.V.'s of type* II *which contain* $(GL(1)^2 \times SL(2) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3)$ *as their component are given by*

$$\big(GL(1)^3 \times SL(2) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes \Lambda_1^*\big) \tag{2.1}$$

and

(2.2) $(GL(1)^{2+s} \times SL(2) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_3)$ *where* $(GL(1)^s \times SL(2),\ \sigma_1 + \cdots + \sigma_s)$ *is a simple P.V., i.e.,* $0 \leqslant s \leqslant 3$; $s = 1$, $\sigma_1 = \Lambda_1, 2\Lambda_1, 3\Lambda_1$; $s = 2$, $\sigma_1 + \sigma_2 = \Lambda_1 + \Lambda_1, 2\Lambda_1 + \Lambda_1$; $s = 3$, $\sigma_1 + \sigma_2 + \sigma_3 = \Lambda_1 + \Lambda_1 + \Lambda_1$.

PROOF. By dimension reasons, we have $k = 1$, i.e., $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + \rho_2 \otimes \Lambda_1$ $(\rho_2 \neq 1)$ is not a P.V. If $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes (\tau_1 + \cdots + \tau_t)$ is a P.V., then we have $t = 1$ and $\tau_1 = \Lambda_1$ or Λ_1^* also by dimension reasons. The $\mathfrak{sl}(6)$-part of the generic isotropy subalgebra $\mathfrak{h}$ of $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3$ at $((I_2 0 \mid I_2 0),\ e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6)$ is given by

$$\mathfrak{h} = \left\{ \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right) \mid A_1 = \left(\begin{array}{c|c} A & B \\ \hline 0 & -\mathrm{Tr} A \end{array} \right), A_2 = \left(\begin{array}{c|c} A & C \\ \hline 0 & -\mathrm{Tr} A \end{array} \right),\ A \in \mathfrak{gl}(2);\ B, C \in K^2 \right\}.$$

Hence, the standard action of $\mathfrak{h}$ on K^6 is nonprehomogeneous since $f(x) = x_3 x_6^{-1}$ for $x = \sum x_i e_i \in K^6$ is a nonconstant absolute invariant. However, its dual action is prehomogeneous. For example, $e_1 + e_5$ is a generic point. In this case, i.e., $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes \Lambda_1^*$, the $SL(2)$ part of the generic isotropy subgroup is $\{1\}$, and hence $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes \Lambda_1^* + \sigma \otimes 1$ is a non-P.V. for any $\sigma \neq 1$. Since $(GL(1) \times SL(6),\ \Lambda_1 \otimes (\Lambda_3 + \Lambda_1 + \Lambda_1))$ is a P.V. (see p. 100 in [**2**]), $(GL(1)^2 \times \{I_2\} \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3)$ is a P.V. and hence we obtain the latter part. Q.E.D.

THEOREM 2.5. *All nonirreducible 2-simple P.V.'s of type* II *which contain* $(GL(1)^2 \times SL(4) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3)$ *as their component are given by*

(2.3) $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + \sigma \otimes 1$ *where* $\sigma = \Lambda_1^*, \Lambda_2^{(*)}$ *or* $\Lambda_1^* + \Lambda_1^*$,

(2.4) $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes \Lambda_1$.

PROOF. By the proof of Lemma 2.3, $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + (\sigma_1 + \cdots + \sigma_s) \otimes 1$ is a P.V. if and only if

$$\left(GL(1)^{1+s} \times SL(4),\ \Lambda_1 \otimes \left(\Lambda_1^* + \Lambda_1^*\right) + (\sigma_1 + \cdots + \sigma_s)\right)$$

is a P.V. and hence $\sigma_1 + \cdots + \sigma_s$ is one of $\Lambda_1^*, \Lambda_2^{(*)}, \Lambda_1^* + \Lambda_1^*$ (see §3 in [**2**]). Note that

$$\left(GL(1)^2 \times SL(4),\ \Lambda_1 \otimes 1 \otimes \left(\Lambda_1^* + \Lambda_1^*\right) + 1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(4)^* + V(4)^* + V(4)\right)$$

is not a P.V. since $f(x) = \langle x_1, y \rangle \cdot \langle x_2, y \rangle^{-1}$ for $x = (x_1, x_2, y) \in V(4)^* + V(4)^* + V(4)$ is a nonconstant absolute invariant. Now $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_3 + \tau_2 + \cdots + \tau_t)$ is a castling transform of

$$\left(GL(1)^{2+t} \times SL(2) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_3 + \tau_2^* + \cdots + \tau_t^*\right)\right),$$

by Theorem 2.4, we have $t = 2$ and $\tau_2 = \Lambda_1$. By the reasons above and dimension reasons, if $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1$ is a P.V., then $\sigma_1 + \cdots + \sigma_s$ must be Λ_1^*. Now

$$\left(GL(1)^4 \times SL(4) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes \Lambda_1 + \Lambda_1^* \otimes 1\right)$$

is isotropic P.V.-equivalent to

$$\left(GL(1)^3 \times SL(4) \times \left(\begin{array}{c|c} SL(3) & 0 \\ \hline 0 & SL(3) \end{array}\right), \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 + \Lambda_1^* \otimes 1\right)$$

which is castling-equivalent to

$$\left(GL(1)^3 \times SL(4) \times \left(\begin{array}{c|c} SL(2) & 0 \\ \hline 0 & SL(2) \end{array}\right), \Lambda_1^* \otimes \Lambda_1 + 1 \otimes \Lambda_1 + \Lambda_1^* \otimes 1\right).$$

Similarly as Proposition 1.12, it is P.V.-equivalent to

$$\left(GL(1)^2 \times \left(\begin{array}{c|c} SL(2) & 0 \\ \hline 0 & SL(2) \end{array}\right), \Lambda_1 + \Lambda_1^*\right).$$

The action is given by $X \mapsto (\alpha Ax, \alpha By, \beta {}^t A^{-1} z, \beta {}^t B^{-1} w)$ for $X = (x, y, z, w) \in V = K^2 + K^2 + K^2 + K^2$ and $g = (\alpha, \beta;\ A, B) \in GL(1)^2 \times SL(2) \times SL(2)$. Since $f(X) = ({}^t xz) \cdot ({}^t yw)^{-1}$ is a nonconstant absolute invariant, it is not a P.V. Thus $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_3 + \Lambda_1) + \Lambda_1^* \otimes 1$ is a non-P.V. Q.E.D.

THEOREM 2.6. *$(GL(1)^3 \times Sp(2) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + \rho \otimes \rho')$ is a P.V. if and only if $\rho = \rho' = 1$.*

PROOF. By dimension reasons, we have $\deg \rho \otimes \rho' \leqslant 4$, and hence $\rho' = 1$ and $\rho = \Lambda_1$ (or 1). In this case, it is P.V.-equivalent to $(GL(1)^4 \times SL(4) \times SL(6),$ $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + (\Lambda_2 + \Lambda_1) \otimes 1)$, which is a non-P.V. by Theorem 2.5. Q.E.D.

LEMMA 2.7. *$(GL(1)^3 \times SL(5) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + \Lambda_2^* \otimes 1$ (resp. $\Lambda_2 \otimes 1$)) is a P.V. (resp. a non-P.V.).*

PROOF. First we shall prove $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + \Lambda_2^* \otimes 1$ is a P.V. By Proposition 1.33, it is P.V.-equivalent to

$$\begin{aligned}&(Sp(2) \times GL(5) \times (GL(1) \times SL(3) \times SL(3)),\\ &\qquad \Lambda_1 \otimes \Lambda_1^* \otimes (1 \otimes 1 \otimes 1) + 1 \otimes \Lambda_1 \otimes (\Lambda_1 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1))\end{aligned}$$

and by successive transformations, we have

$$\begin{aligned}&(Sp(2) \times GL(5) \times (GL(1) \times SL(2) \times SL(2)),\\ &\qquad \Lambda_1 \otimes \Lambda_1^* \otimes (1 \otimes 1 \otimes 1) + 1 \otimes \Lambda_1^* \otimes (\Lambda_1 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1))\\ &\sim (Sp(2) \times GL(3) \times (GL(1) \times SL(2) \times SL(2)),\\ &\qquad \Lambda_1 \otimes \Lambda_1 \otimes (1 \otimes 1 \otimes 1) + 1 \otimes \Lambda_1 \otimes (\Lambda_1 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1))\\ &\sim (Sp(2) \times GL(3) \times GL(1),\ \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes (\Lambda_1 + \Lambda_1)).\end{aligned}$$

Since $Sp(1) = SL(2)$, similarly as Proposition 1.33, this is P.V.-equivalent to

$$(SL(2) \times GL(3) \times GL(1),\ \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes (\Lambda_1 + \Lambda_1))$$

which is castling-equivalent to a trivial P.V. $(GL(3) \times GL(1), \Lambda_1 \otimes (1 + \Lambda_1 + \Lambda_1))$. Next we shall prove that $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + \Lambda_2 \otimes 1$ is a non-P.V. By Proposition 1.33, it is P.V.-equivalent to

$$\big(Sp(2) \times (GL(1) \times SL(3) \times SL(3)) \times GL(5),\\ \Lambda_1 \otimes 1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1\big).$$

From a castling transform

$$\big(Sp(2) \times (GL(1) \times SL(2) \times SL(2)) \times GL(5),\\ (\Lambda_1 \otimes 1 \otimes 1 \otimes 1) \otimes \Lambda_1 + \{1 \otimes \Lambda_1 \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1)\} \otimes \Lambda_1^*\big),$$

we obtain the P.V.-equivalence to

$$(GL(1) \times Sp(2) \times SL(2) \times SL(2), \Lambda_1 \otimes (\Lambda_1 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1))$$

by Theorems 1.14 and 1.16. However it is a non-P.V. since

$$(Sp(2) \times GL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1, M(4,2) + M(4,2))$$

has a nonconstant absolute invariant $f(x) = \det({}^tXJX) \cdot \det({}^tYJY) \cdot \det(X \mid Y)^{-2}$ for

$$x = (X, Y) \in M(4,2) + M(4,2) \quad \text{where } J = \left(\begin{array}{c|c} 0 & I_2 \\ \hline -I_2 & 0 \end{array}\right). \quad \text{Q.E.D.}$$

LEMMA 2.8. *$(GL(1)^4 \times SL(5) \times SL(6), \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + (\Lambda_1^* + \Lambda_1^{(*)}) \otimes 1$ (resp. $(\Lambda_1 + \Lambda_1) \otimes 1$)) are P.V.'s (resp. is a non-P.V.).*

PROOF. First consider $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + (\Lambda_1^* + \Lambda_1^*) \otimes 1$. It is P.V.-equivalent to

$$\big(GL(1)^3 \times SL(5) \times (SL(3) \times SL(3)),\\ \Lambda_1 \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) + (\Lambda_1^* + \Lambda_1^*) \otimes 1 \otimes 1\big).$$

By successive castling transformations, we get

$$\big(GL(1)^3 \times SL(5) \times (SL(2) \times SL(2)),\\ \Lambda_1^* \otimes ([\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1] + 1 \otimes 1 + 1 \otimes 1)\big)\\ \sim \big(GL(1)^3 \times (SL(2) \times SL(2)), (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) + 1 \otimes 1 + 1 \otimes 1\big),$$

which is P.V.-equivalent to a trivial P.V. $(GL(1) \times SL(2) \times SL(2), \Lambda_1 \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1))$. Next consider $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + (\Lambda_1^* + \Lambda_1) \otimes 1$. Since the $GL(5)$-part of a generic isotropy subgroup of $(GL(1)^2 \times GL(5), \Lambda_1^* + \Lambda_1)$ is

$$\left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & \alpha \end{array}\right) \Bigm| A \in GL(4),\ \alpha \in GL(1)\right\},$$

it is P.V.-equivalent to

$$(GL(4) \times GL(1) \times SL(3) \times SL(3),$$
$$\Lambda_1 \otimes 1 \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) + 1 \otimes \Lambda_1 \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1)),$$

which is castling-equivalent to a generalized direct sum

$$(GL(4) \times GL(1) \times SL(2) \times SL(2),$$
$$\Lambda_1 \otimes (1 \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1) + (1 \otimes \Lambda_1 \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1)))$$

of trivial P.V.'s. Finally we shall prove that $(\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + (\Lambda_1 + \Lambda_1) \otimes 1)$ is a non-P.V. By a castling transformation, we get

$$\left(GL(1)^3 \times SL(3) \times SL(6),\ \Lambda_1 \otimes \Lambda_1^* + 1 \otimes \Lambda_3 + (\Lambda_1 + \Lambda_1) \otimes 1\right)$$
$$\simeq \left(GL(1)^3 \times SL(3) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + (\Lambda_1 + \Lambda_1) \otimes 1\right),$$

which is a non-P.V. by Lemma 2.3. Q.E.D.

THEOREM 2.9. *All nonirreducible 2-simple P.V.'s of type* II *which contain* $(GL(1)^2 \times SL(5) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3)$ *as their component are given by*

(2.5) $\quad \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + \sigma \otimes 1$ *where* $\sigma = \Lambda_1^{(*)},\ \Lambda_2^*,\ \Lambda_1^* + \Lambda_1^{(*)}$;

(2.6) $\quad \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes \Lambda_1^*$.

PROOF. By dimension reasons, $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + \rho \otimes \rho'$ with $\rho \neq 1$ and $\rho' \neq 1$, is a non-P.V. Assume that $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + (\sigma_1 + \cdots + \sigma_s) \otimes 1$ is a P.V. Then we have $11 + s \geqslant \deg(\sigma_1 + \cdots + \sigma_s) \geqslant 5s$ and hence $s = 1$ or 2. By dimension reasons, we have $\sigma_1 = \Lambda_1^{(*)},\ \Lambda_2^{(*)}$ and $\sigma_1 + \sigma_2 = \Lambda_1^{(*)} + \Lambda_1^{(*)}$. By Lemmas 2.7 and 2.8, we have $\sigma_1 = \Lambda_1^{(*)},\ \Lambda_2^*$ and $\sigma_1 + \sigma_2 = \Lambda_1^* + \Lambda_1^{(*)}$. Now assume that $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_3 + \tau_2 + \cdots + \tau_t)$ $(t \geqslant 2)$ is a P.V. It is castling-equivalent to $(GL(1)^{t+1} \times SL(6),\ \Lambda_3 + \Lambda_1^* + \tau_2 + \cdots + \tau_t)$ and hence, we have $t = 2$ and $\tau_2 = \Lambda_1^*$. Assume that $(GL(1)^4 \times SL(5) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes \Lambda_1^* + \sigma_1 \otimes 1)$ is a P.V. Then, by dimension reasons, σ_1 must be Λ_1 or Λ_1^*. If $\sigma_1 = \Lambda_1^*$, then $(GL(1)^3 \times SL(6),\ \Lambda_3 + \Lambda_1 + \Lambda_1^*)$ must be a P.V. by Theorem 1.14, which is a contradiction (see p. 86 in [2]). If $\sigma_1 = \Lambda_1$, we have a castling transform $(GL(1)^4 \times SL(2) \times SL(6),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 + 1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$, which is a non-P.V. by Theorem 2.4. Q.E.D.

Now let us consider the case when $n = 2m + 1$ and $\tau_1 = \Lambda_2^{(*)}$.

LEMMA 2.10. *Let G be a simple algebraic group and assume that $2 \leqslant \deg \rho_i \leqslant 2m$ for $i = 1, \ldots, k$. Then $(GL(1)^{1+k} \times G \times SL(2m+1),\ (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + 1 \otimes \Lambda_2)$ $(m \geqslant 2)$ is a P.V. if and only if $k = 1$ and $(G, \rho_1) = (SL(n'), \Lambda_1^{(*)}), (Sp(m'), \Lambda_1), (SL(2), 2\Lambda_1) \simeq (SO(3), \Lambda_1)$.*

PROOF. Put $n = \deg \rho_1 + \cdots + \deg \rho_k$. First consider the case for $n \leqslant 2m + 1$. By (2) of Proposition 1.34, it is P.V.-equivalent to

$$\left(GL(1)^{1+k} \times G \times SL(n-1),\ (\rho_1^* + \cdots + \rho_k^*) \otimes \Lambda_1 + 1 \otimes \Lambda_2\right).$$

In particular, we have $k + \dim G \geqslant \frac{1}{2}(n^2 - n)$. Note that $(GL(1)^k \times G \times SL(n-1), (\rho_1^* + \cdots + \rho_k^*) \otimes \Lambda_1)$ is castling-equivalent to $(GL(1)^k \times G, \rho_1 + \cdots + \rho_k, V(n))$. Hence, by **[2]**, we have $k = 1$ and (G, ρ_1) must be one of $(SL(n), \Lambda_1^{(*)})$, $(Sp(m'), \Lambda_1)$, $(SO(n), \Lambda_1)$. By [3] and Definition 1.10, $\rho_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2$ is actually a P.V. if and only if (G, ρ_1) is one of $(SL(n), \Lambda_1^{(*)})$, $(Sp(m'), \Lambda_1)$, or $(SO(3), \Lambda_1) = (SL(2), 2\Lambda_1)$. Next consider the case for $n \geqslant 2m + 2$. Then we have $k \geqslant 2$ and by the discussion above,

$$\left(GL(1)^3 \times SL(n') \times SL(2m+1), \left(\Lambda_1 + \Lambda_1^{(*)}\right) \otimes \Lambda_1 + 1 \otimes \Lambda_2\right) \qquad (n = 2n' \geqslant 2m + 2)$$

must be a P.V. In particular,

$$1 + n'^2 + (2m+1)^2 \geqslant 2n'(2m+1) + m(2m+1),$$

and hence $(2m + 1 - n')^2 \geqslant 2m^2 + m - 1$. Since $2m + 1 > n' \geqslant m + 1$, we have $2m + 1 - \sqrt{2m^2 + m - 1} > n' \geqslant m + 1$. This implies $m(m+1) \leqslant 1$ with $m \geqslant 2$, which is a contradiction. Q.E.D.

LEMMA 2.11. *Let G be a simple algebraic group and assume that $2 \leqslant \deg \rho_i \leqslant 2m$ for $i = 1, \ldots, k$. Then*

$$\left(GL(1)^{1+k} \times G \times SL(2m+1), (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + 1 \otimes \Lambda_2^*\right) \qquad (m \geqslant 2)$$

is a P.V. if and only if $k = 1$ and $(G, \rho_1) = (SL(n'), \Lambda_1^{(*)}), (SL(2), 2\Lambda_1)$.

PROOF. First assume that $k = 1$. By (1) of Proposition 1.34, it is P.V.-equivalent to $(GL(1) \times G \times Sp(m), \rho_1 \otimes \Lambda_1)$ $(m \geqslant 2)$ and hence $(G, \rho_1) = (SL(n'), \Lambda_1^{(*)})$ or $(SL(2), 2\Lambda_1) \simeq (SO(3), \Lambda_1)$. Now assume that it is a P.V. with $k \geqslant 2$. Then, by (1) of Proposition 1.34,

$$\begin{aligned}&\left(GL(1)^2 \times G \times Sp(m), (\rho_1 + \rho_2) \otimes \Lambda_1\right)\\ &\qquad = \left(GL(1)^2 \times G \times Sp(m), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*\right)\end{aligned}$$

must be a P.V., which is a contradiction by Proposition 1.15. Q.E.D.

THEOREM 2.12.

$$\begin{aligned}&\Big(GL(1)^{1+s+t} \times G \times SL(2m+1),\\ &\qquad \rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \left(\Lambda_2^* + R_{t-1}\right)\Big) \qquad (2 \leqslant \deg \rho_1 \leqslant 2m)\end{aligned}$$

with $R_0 = 0$ (resp. $R_1 = \Lambda_1^{(*)}$, $t = 1,2$) is a P.V. if and only if it is one of the following:

(2.7) $(GL(1)^{2+s} \times SL(2) \times SL(2m+1), 2\Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_2^*)$ with (1) $s = 0$, (2) $s = 1$, $\sigma_1 = \Lambda_1$;

(2.8) $(GL(1)^{2+s} \times SL(n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_2^*)$ $(2 \leqslant n' \leqslant 2m)$ with (1) $s = 0$, (2) $s = 1$, $\sigma_1 = \Lambda_1^{(*)}$, Λ_2 $(n' = \text{odd})$, $2\Lambda_1^{(*)}$ $(n' = 2, 3)$, $3\Lambda_1$ $(n' = 2)$, (3) $s = 2$; $\sigma_1 + \sigma_2 = \Lambda_1^{(*)} + \Lambda_1^{(*)}$, $\Lambda_2 + \Lambda_1^*$ $(n' = 5)$, $\Lambda_1 + 2\Lambda_1$ $(n' = 2)$, (4) $s = 3$; $\sigma_1 + \sigma_2 + \sigma_3 = \Lambda_1 + \Lambda_1 + \Lambda_1$, $\Lambda_1 + \Lambda_1^* + \Lambda_1^*$, $\Lambda_1^* + \Lambda_1^* + \Lambda_1^*$, $\Lambda_1 + \Lambda_1 + \Lambda_1^*$ $(n' = \text{even})$;

(2.9) $(GL(1)^{3+s} \times SL(n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\Lambda_2^* + \Lambda_1^{(*)}))$ $(2 \leqslant n' \leqslant 2m)$ with (1) $s = 0$, (2) $s = 1$; $\sigma_1 = \Lambda_1^{(*)}$, $2\Lambda_1$ $(n' = 2)$, (3) $s = 2$; $\sigma_1 + \sigma_2 = (\Lambda_1 + \Lambda_1)^{(*)}$, $\Lambda_1 + \Lambda_1^*$ $(n' = \text{even})$.

PROOF. Similarly as the proof of Lemma 2.11, we have our results. Q.E.D.

THEOREM 2.13. *Let us consider*

$$\left(GL(1)^{2+s} \times G \times SL(2m+1),\ \rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_2\right) \quad (2 \leqslant \deg \rho_1 \leqslant 2m).$$

(2.10) *If* $(G, \rho_1) = (SL(2n'+1), \Lambda_1)$, *then it is a P.V. if and only if* $(GL(1)^{1+s} \times SL(2n'+1),\ \Lambda_2^* + \sigma_1 + \cdots + \sigma_s)$ *is a simple P.V., i.e.,* $0 \leqslant s \leqslant 3$; $s = 1$, $\sigma_1 = \Lambda_2^*$, $\Lambda_1^{(*)}$, $2\Lambda_1^{(*)}$ $(n' = 1)$; $s = 2$, $\sigma_1 + \sigma_2 = \Lambda_2^* + \Lambda_1$ $(n' = 2)$, $\Lambda_1^{(*)} + \Lambda_1^{(*)}$; $s = 3$, $\sigma_1 + \sigma_2 + \sigma_3 = (\Lambda_1 + \Lambda_1 + \Lambda_1)^{(*)}$, $\Lambda_1 + \Lambda_1 + \Lambda_1^*$.

(2.11) *If* $(G, \rho_1) = (Sp(m'), \Lambda_1)$, *then it is a P.V. if and only if* $s = 0$; *or* $s = 1$, $m' = 2$, $\sigma_1 = \Lambda_1, \Lambda_2$.

(2.12) *If* $(G, \rho_1) = (SL(2), 2\Lambda_1)$, *then it is a P.V. if and only if* $s = 0$; *or* $s = 1$, $\sigma_1 = \Lambda_1$.

PROOF. By (2) of Proposition 1.34, it is P.V.-equivalent to

$$\left(GL(1)^{2+s} \times G \times SL(\deg \rho_1 - 1),\ \rho_1^* \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_2\right).$$

For (2.10), it is P.V.-equivalent to

$$\left(GL(1)^{1+s} \times SL(2n'+1) \times Sp(n'), \Lambda_1^* \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1\right).$$

By Proposition 1.33, it is P.V.-equivalent to $(GL(1)^{1+s} \times SL(2n'+1),\ \Lambda_2^* + \sigma_1 + \cdots + \sigma_s)$. For (2.11), we have our result by Theorem 2.24 in [3]. For (2.12), since $(GL(1) \times SL(2),\ \Lambda_1 \otimes \Lambda_2) \cong (GL(1), \Lambda_1)$, it is P.V.-equivalent to

$$\left(GL(1)^{1+s} \times SL(2) \times SL(2), 2\Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1\right),$$

which is castling-equivalent to a simple case and we obtain our result. Q.E.D.

LEMMA 2.14. (1)

$$\left(GL(1)^4 \times SL(2n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda_1^*) \otimes 1\right)$$

is a P.V.

(2)

$$\left(GL(1)^5 \times SL(2n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda_1^* + \Lambda_1^*) \otimes 1\right)$$

is a non-P.V.

PROOF. By (2) of Proposition 1.34, (1) is P.V.-equivalent to

$$\left(GL(1)^4 \times SL(2n') \times SL(2n'-1),\ \Lambda_1^* \otimes (\Lambda_1 + 1) + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_2\right).$$

By Proposition 1.12, it is P.V.-equivalent to a simple P.V. $(GL(1)^2 \times SL(2n'-1), \Lambda_2 + \Lambda_1)$. Assume that (2) is a P.V. Then, by Theorem 1.14, $(GL(1)^5 \times SL(2m+1), (\Lambda_1 + 1) \otimes (1 + 1) + \Lambda_2)$ must be a P.V. This implies the prehomogeneity of the triplet

$$(GL(1) \times GL(1) \times SL(2m+1),\ \Lambda_1 \otimes 1 \otimes \Lambda_2 + 1 \otimes \Lambda_1 \otimes (\Lambda_1 + \Lambda_1), V)$$

where

$$V = \left\{(X;\ y, z) \mid {}^tX = -X \in M(2m+1),\ y,\ z \in K^{2m+1}\right\}.$$

However a rational function $f(x) = f_1(x) f_2(x)^{-1}$ is a nonconstant absolute invariant where

$$f_1(x) = Pf\left(\begin{array}{c|c} X & y \\ \hline -{}^ty & 0 \end{array}\right) \quad \text{and} \quad f_2(x) = Pf\left(\begin{array}{c|c} X & z \\ \hline -{}^tz & 0 \end{array}\right),$$

which is a contradiction. Q.E.D.

LEMMA 2.15. (1)

$$\left(GL(1)^5 \times SL(2n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda_1 + \Lambda_1) \otimes 1\right)$$

is a non-P.V. (resp. P.V.) if $n' = 1$ *(resp.* $2 \leqslant n' \leqslant m$*).*

(2)

$$\left(GL(1)^6 \times SL(2n') \times SL(2m+1),\right.$$
$$\left.\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda_1 + \Lambda_1 + \Lambda_2) \otimes 1\right)$$

is a non-P.V.

PROOF. By (2) of Proposition 1.34, (1) (resp. (2)) is P. V.-equivalent to

$$\left(GL(1)^5 \left(\text{resp. } GL(1)^6\right) \times SL(2n') \times SL(2n'-1),\right.$$
$$\left.\Lambda_1^* \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda_1 + \Lambda_1) \otimes 1 \left(\text{resp. } (\Lambda_1 + \Lambda_1 + \Lambda_1 + \Lambda_1) \otimes 1\right)\right).$$

If $n' = 1$, it is P.V.-equivalent to

$$\left(GL(1)^4 \left(\text{resp. } GL(1)^5\right) \times SL(2),\right.$$
$$\left.\Lambda_1 + \Lambda_1 + \Lambda_1 + \Lambda_1 \left(\text{resp. } \Lambda_1 + \Lambda_1 + \Lambda_1 + \Lambda_1 + \Lambda_1\right)\right),$$

which is a non-P.V. by dimension reasons. If $n' \geqslant 2$, by Theorems 1.14 and 1.16 it is P.V.-equivalent to

$$\left(GL(1)^4 \left(\text{resp. } GL(1)^5\right) \times SL(2n'-1),\right.$$
$$\left.\Lambda_2 + \Lambda_1 + \Lambda_1 + \Lambda_1 \left(\text{resp. } \Lambda_2 + \Lambda_1 + \Lambda_1 + \Lambda_1 + \Lambda_1\right)\right),$$

which is a P.V. (resp. a non-P.V.) by Theorem 1.3 in [**3**]. Q.E.D.

THEOREM 2.16.

$$\left(GL(1)^{2+s}\times SL(2n')\times SL(2m+1),\ \Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2+(\sigma_1+\cdots+\sigma_s)\otimes 1\right)$$
$$(1\leqslant n'\leqslant m, s\geqslant 1)$$

is a P.V. if and only if $s=0$ or it is one of the following cases.

(2.13) $s=1;\ \sigma_1=\Lambda_1^{(*)},\ \Lambda_2^{(*)}\ (2\leqslant n'\leqslant m), 2\Lambda_1^{(*)}\quad (n'=1,2)$,

(2.14) $s=2;\ \sigma_1+\sigma_2=\Lambda_1^{(*)}+\Lambda_1^{(*)},\ \Lambda_2+\Lambda_1^{(*)}\quad (n'=2)$,

(2.15) $s=3;\ \sigma_1+\sigma_2+\sigma_3=(\Lambda_1+\Lambda_1+\Lambda_1)^{(*)}\quad (2\leqslant n'\leqslant m)$.

PROOF. First consider the case when $\sigma_1+\cdots+\sigma_s\neq\Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)}$. We may assume that $\sigma_1\neq\Lambda_1^{(*)}$. By (2) of Proposition 1.34, it is P.V.-equivalent to

$$\left(GL(1)^{2+s}\times SL(2n')\times SL(2n'-1),\ \Lambda_1^*\otimes\Lambda_1+1\otimes\Lambda_2+(\sigma_1+\cdots+\sigma_s)\otimes 1\right).$$

Hence if $n'=1$, we have $s=1$ and $\sigma_1=2\Lambda_1$ by Theorem 1.3 in [3]. Now assume that $n'\geqslant 2$. If $\sigma_1=2\Lambda_1^{(*)}$, then

$$\left(GL(1)^{1+s}\times SO(2n')\times SL(2n'-1),\ \Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2+(\sigma_2+\cdots+\sigma_s)\otimes 1\right)$$

must be a P.V. By Theorem 2.5 in [3], it is a non-P.V. for $n'\geqslant 3$. If $n'=2$, we have $s=1$ by dimension reason. If $\sigma_1=\Lambda_3^{(*)}$ $(n'=3,4)$, it is a non-P.V. by Proposition 2.2 and Lemma 2.7. If $\sigma_1=\Lambda_2^{(*)}$, then

$$\left(GL(1)^{1+s}\times Sp(n')\times SL(2m+1),\ \Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2+(\sigma_2+\cdots+\sigma_s)\otimes 1\right)$$

must be a P.V. By (2.11) in Theorem 2.13, we have $s=1$, or $s=2$, $n'=2$, $\sigma_2=\Lambda_1^{(*)},\ \Lambda_2$. However $\sigma_2\neq\Lambda_2$ since $(GL(1)^2\times SL(2n'),\ \Lambda_2^{(*)}+\Lambda_2^{(*)})$ is a non-P.V. Finally consider the case when $\sigma_1+\cdots+\sigma_s=\Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)}$. By Lemmas 2.14 and 2.15 and Proposition 1.37, we obtain our result. Q.E.D.

THEOREM 2.17.

$$\left(GL(1)^{3+s}\times G\times SL(2m+1), \rho_1\otimes\Lambda_1+(\sigma_1+\cdots+\sigma_s)\otimes 1+1\otimes(\Lambda_2+\Lambda_1)\right)$$
$$(2\leqslant \deg\rho_1\leqslant 2m)$$

is a P.V. if and only if $(G,\rho_1)=(SL(n'),\Lambda_1)$ and $(GL(1)^{2+s}\times SL(n'),\ \Lambda_2+\Lambda_1+\sigma_1+\cdots+\sigma_s)$ is a simple P.V., i.e., $0\leqslant s\leqslant 2$; $s=1$, $\sigma_1=\Lambda_1^{(*)},\ 2\Lambda_1\ (n'=2)$; $s=2$, $\sigma_1+\sigma_2=\Lambda_1^{(*)}+\Lambda_1^{(*)}$ except $\Lambda_1+\Lambda_1^*$ for $n'=odd$.

PROOF. By (2) of Proposition 1.34, it is P.V.-equivalent to

$$\left(GL(1)^{3+s}\times G\times SL(n'), \rho_1^*\otimes\Lambda_1+1\otimes(\Lambda_1+\Lambda_2)+(\sigma_1+\cdots+\sigma_s)\otimes 1\right)$$

with $n'=\deg\rho_1$. When $(G,\rho_1)=(SL(n'),\ \Lambda_1)$, it is P.V.-equivalent to $(GL(1)^{2+s}\times SL(n'),\ \Lambda_2+\Lambda_1+\sigma_1+\cdots+\sigma_s)$ by Proposition 1.12. If $(G,\rho_1)=(Sp(m'),\Lambda_1)$ or $(SL(2), 2\Lambda_1)$, then $(GL(1)^3\times G\times SL(n'),\ \rho_1^*\otimes\Lambda_1+1\otimes(\Lambda_1+\Lambda_2))$ $(n'=2m'$ or $n'=3)$ is a non-P.V. Actually by Proposition 13 (resp. Proposition 14) in §2 in [1], it is P.V.-equivalent to $(GL(1)^3\times SL(2m'),\ \Lambda_2+\Lambda_2+\Lambda_1)$ (resp. $(GL(1)^3\times SL(3), 2\Lambda_1+\Lambda_1+\Lambda_1^*)$) which is a non-P.V. by [2]. Q.E.D.

LEMMA 2.18. (1)

$$\left(GL(1)^5 \times SL(2n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1) \otimes 1 + 1 \otimes \left(\Lambda_2 + \Lambda_1^*\right)\right)$$

is a P.V. for $2 \leqslant n' \leqslant m$.

(2)

$$\left(GL(1)^4 \times SL(2n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes \left(\Lambda_2 + \Lambda_1^*\right)\right)$$

is a P.V. for $1 \leqslant n' \leqslant m$.

PROOF. (1) is castling-equivalent to

$$\left(GL(1)^5 \times SL(2m+3-2n') \times SL(2m+1),\ (\Lambda_1 + 1) \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + (\Lambda_1 + \Lambda_1) \otimes 1\right).$$

Since $(2m+3-2n')+1 \leqslant 2m$, by (1) of Proposition 1.34, it is P.V.-equivalent to

$$\left(GL(1)^4 \times SL(2m+3-2n') \times Sp(m), \Lambda_1 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1) \otimes 1 + 1 \otimes \Lambda_1\right),$$

which is a P.V. by (2.78) in [**3**]. By the same method as the proof of Lemma 2.20, one can easily check that (2) is P.V.-equivalent to $(GL(1)^2 \times SL(2n'-1) \times Sp(m),$ $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$, which is actually a P.V. by (2.76) in [**3**]. Q.E.D.

THEOREM 2.19.

$$\left(GL(1)^{3+s} \times G \times SL(2m+1),\ \rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \left(\Lambda_2 + \Lambda_1^*\right)\right) \qquad (2 \leqslant \deg \rho_1 \leqslant 2m)$$

is a P.V. if and only if it is one of the following:

(2.16) $(GL(1)^{3+s} \times SL(n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\Lambda_2 + \Lambda_1^*))$ $(2 \leqslant n' \leqslant 2m)$ *with* (1) $s = 0$, (2) $s = 1$; $\sigma_1 = \Lambda_1^{(*)}$; $\sigma_1 = \Lambda_2$ $(n' = 4)$; $\sigma_1 = \Lambda_2^*$ $(n' = 5)$, (3) $s = 2$; $\sigma_1 + \sigma_2 = \Lambda_1 + \Lambda_1$ $(n' \geqslant 3)$; $\sigma_1 + \sigma_2 = \Lambda_1 + \Lambda_1^*$ $(n' = \text{odd})$;

(2.17) $\left(GL(1)^3 \times Sp(2) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2 + \Lambda_1^*\right)\right)$.

PROOF. First assume that $n' = \text{odd}$. By Proposition 1.36, it is P.V.-equivalent to $(GL(1)^{2+s} \times G,\ \Lambda^2(\rho_1)^* + \rho_1 + \sigma_1 + \cdots + \sigma_s)$. By Theorem 1.3 in [**3**], it is a P.V. if and only if $(G, \rho_1, \sigma_1, \ldots, \sigma_s) = (SL(n'), \Lambda_1)$, $(SL(n'), \Lambda_1, \Lambda_1^{(*)})$, $(SL(n'),$ $\Lambda_1, \Lambda_1^{(*)}, \Lambda_1)$, or $(SL(5), \Lambda_1, \Lambda_2^*)$. By this result for $(SL(5), \Lambda_1, \Lambda_2^*)$ and by Proposition 1.33,

$$\left(GL(1)^4 \times Sp(2) \times SL(2m+1) \times SL(5),\ 1 \otimes \left(\Lambda_2 + \Lambda_1^*\right) \otimes 1 + (1 \otimes \Lambda_1) \otimes \Lambda_1 + (\Lambda_1 \otimes 1) \otimes \Lambda_1^*\right)$$

is a P.V. Hence by Theorem 1.14, (2.17) is a P.V. Hence (2.16) for $\sigma_1 = \Lambda_2$, $n' = 4$ is also a P.V. Next assume that $n' =$ even. By Proposition 1.35,

$$\left(GL(1)^{3+s} \times G \times SL(2m+1),\ \rho_1 \otimes \Lambda_1 + (\rho_1 + \sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_2\right)$$

must be a P.V. Hence, by Lemma 2.10 and Theorems 2.13 and 2.16, we have (2.16) and (2.17). By Lemma 2.18, (2.16) for $\sigma_1 = \Lambda_1^{(*)}$, or $\sigma_1 + \sigma_2 = \Lambda_1 + \Lambda_1$ ($n' \geqslant 4$) is actually a P.V. Q.E.D.

LEMMA 2.20. *If* $n =$ odd *or* $n = 2$, *then*

$$\left(GL(1)^5 \times SL(n) \times SL(2m+1),\right.$$
$$\left.\Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + 1 \otimes \Lambda_1^{(*)} + 1 \otimes \Lambda_1^{(*)}\right)$$

is a non-P.V.

PROOF. Since a generic isotropy subalgebra of $(GL(n), \Lambda_1^*)$ ($(GL(2m+1), \Lambda_2)$, respectively) is

$$\left\{\left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & \alpha \end{array}\right);\ A_1 \in \mathfrak{gl}(n-1)\right\} \qquad \left(\text{resp. } \left\{\left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & \beta \end{array}\right);\ B_1 \in \mathfrak{sp}(m)\right\}\right)$$

and

$$\gamma\left(\begin{array}{c|c} X & 0 \\ \hline 0 & 1 \end{array}\right) + \left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & \alpha \end{array}\right)\left(\begin{array}{c|c} X & 0 \\ \hline 0 & 1 \end{array}\right) + \left(\begin{array}{c|c} X & 0 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} {}^tB_1 & 0 \\ \hline {}^tB_2 & \beta \end{array}\right)$$
$$= \left(\begin{array}{c|c} \gamma X + A_1 X + X\,{}^tB_1 & A_2 \\ \hline {}^tB_2 & \gamma + \alpha + \beta \end{array}\right)$$

where X is a generic point of

$$(GL(1) \times SL(n-1) \times Sp(m),\ \Lambda_1 \otimes \Lambda_1,\ M(n-1, 2m)),$$

it is P.V.-equivalent to

$$a^{-1}b^{-1}A \otimes B + \delta\left(\begin{array}{c|c} B & 0 \\ \hline 0 & b \end{array}\right)^{(*)} + \varepsilon\left(\begin{array}{c|c} B & 0 \\ \hline 0 & b \end{array}\right)^{(*)},$$

i.e.,

$$a^{-1}b^{-1}A \otimes B + b^{\pm}B + b^{\pm}B \qquad (A \in GL(n-1),\ B \in Sp(m)).$$

By Lemma 2.20 in [3], it is a non-P.V. if $n - 1 =$ even, i.e., $n =$ odd. If $n = 2$, then it is $a^{-1}b^{-1}B + b^{\pm}B + b^{\pm}B$ ($B \in Sp(m)$) which is a non-P.V. because of the dependence of scalar multiplications (see p. 100 in [2]). Q.E.D.

LEMMA 2.21. *Assume that*

$$\left(GL(1)^{4+s} \times G \times SL(2m+1),\right.$$
$$\left.\rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1^{(*)}\right)$$

is a P.V. Then we have $(G, \rho_1) = (SL(m'), \Lambda_1)$. *Moreover if* $s \geqslant 1$ *and* $\sigma_1 \neq 1$, *then we have* $s = 1$ *and* $\sigma_1 = \Lambda_1^{(*)}$.

PROOF. Since $(GL(1)^2 \times SL(2m+1), \Lambda_2 + \Lambda_1)$ is a regular P.V., we may only consider the case for $\rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_2^* + 1 \otimes \Lambda_1^* + 1 \otimes \Lambda_1^{(*)}$ by Remark 1.32. By Proposition 1.33, $\rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_2^* + 1 \otimes \Lambda_1^* + 1 \otimes \Lambda_1$ is P.V.-equivalent to

$$\begin{aligned}\big(GL(1)^{4+s} \times G \times SL(2m+1) \times Sp(m),\\ (\sigma_1 + \cdots + \sigma_s) \otimes 1 \otimes 1 + (\rho_1 + 1) \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes (\Lambda_1 + 1)\big).\end{aligned}$$

By Theorem 1.14

$$\big(GL(1)^{3+s} \times G \times Sp(m), (\sigma_1 + \cdots + \sigma_s + \rho_1) \otimes 1 + \rho_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1\big)$$

must be a P.V. Then, by Theorem 2.5 in [**3**] and Lemma 2.11, we have $(G, \rho_1) = (SL(m'), \Lambda_1^{(*)})$. Now assume that $s \geqslant 1$ and $\sigma_1 \neq 1$. By (2.75) and (2.78) in [**3**], we have $s = 1$, $\sigma_1 = \Lambda_1^{(*)}$, and we may assume $(G, \rho_1) = (SL(m'), \Lambda_1)$. Similarly, if $\rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\Lambda_2^* + \Lambda_1^* + \Lambda_1^*)$ is a P.V., then

$$\big(GL(1)^{3+s} \times G \times Sp(m), \rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s + \rho_1 + \rho_1) \otimes 1\big)$$

must be a P.V., and we have our result. Q.E.D.

LEMMA 2.22. (1)

$$\big(GL(L)^4 \times SL(m') \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1\big)$$

is a P.V. if and only if $m' =$ odd.

(2)

$$\big(GL(1)^4 \times SL(m') \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1^*\big)$$

is a P.V. if and only if $m' =$ even.

PROOF. Note that $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + 1 \otimes \Lambda_1^{(*)}$ is reductive P.V.-equivalent to $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^* + \Lambda_1^*) + 1 \otimes \Lambda_1^{(*)}$. First assume that $m' \leqslant 2m - 1$. Then, by (2) of Proposition 1.34, $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + 1 \otimes \Lambda_1$ is P.V.-equivalent to

$$\big(GL(1)^4 \times SL(m') \times SL(m'+1), \Lambda_1^* \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1 + \Lambda_1)\big),$$

which is castling-equivalent to $(GL(1)^4 \times SL(m'+1), \Lambda_2 + \Lambda_1^* + \Lambda_1 + \Lambda_1)$. By [**2**], it is a P.V. if and only if $m' + 1 =$ even, i.e., $m' =$ odd. In the case of $m' = 2m$, $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + 1 \otimes \Lambda_1$ is castling-equivalent to

$$\big(GL(1)^4 \times SL(2m+1), \Lambda_1^* + \Lambda_2 + \Lambda_1 + \Lambda_1\big),$$

which is a non-P.V. by [**2**]. For (2), $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^* + 1 \otimes \Lambda_1^*$ is castling-equivalent to

$$\begin{aligned}\big(GL(1)^4 \times SL(2m+1-m') \times SL(2m+1), \Lambda_1 \otimes \Lambda_1^* + 1 \otimes (\Lambda_2^* + \Lambda_1^* + \Lambda_1^*)\big)\\ \cong \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + 1 \otimes \Lambda_1,\end{aligned}$$

and it reduces to the case (1). Hence it is a P.V. if and only if $2m + 1 - m' =$ odd, i.e., $m' =$ even. Q.E.D.

THEOREM 2.23.

$$\left(GL(1)^{4+s} \times G \times SL(2m+1),\ \rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1^{(*)}\right)$$
$$(2 \leqslant \deg \rho_1 \leqslant 2m)$$

is a P.V. if and only if it is one of the following.

(2.18) $(GL(1)^4 \times SL(2n'+1) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1)$ $(1 \leqslant n' \leqslant m-1)$

(2.19) $(GL(1)^4 \times SL(2n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1^*)$ $(1 \leqslant n' \leqslant m)$

PROOF. By Lemma 2.20,

$$\left(GL(1)^5 \times SL(2n'+1) \times SL(2m+1),\ \Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + 1 \otimes \Lambda_1\right)$$

is a non-P.V. and hence so is

$$\Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^* + 1 \otimes \Lambda_1.$$

Now

$$\left(GL(1)^5 \times SL(2n'+1) \times SL(2m+1),\ \Lambda_1 \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1\right)$$

is castling-equivalent to

$$\left(GL(1)^5 \times SL(2m-2n'+1) \times SL(2m+1),\ \Lambda_1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1\right)$$
$$\cong \Lambda_1 \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1^*$$

which is a non-P.V. by (2) of Lemma 2.22. Now

$$\left(GL(1)^5 \times SL(2n') \times SL(2m+1),\ \Lambda_1 \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1^*\right)$$

is castling-equivalent to

$$\left(GL(1)^4 \times SL(2m-2n'+2) \times SL(2m+1),\ \Lambda_1 \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1\right)$$

which is a non-P.V. by (1) of Lemma 2.22. If

$$\left(GL(1)^5 \times SL(2n') \times SL(2m+1),\ \Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1)^{(*)} + 1 \otimes \Lambda_1^*\right)$$

is a P.V., then by Theorem 1.14, $(GL(1)^4 \times SL(2m+1),\ \Lambda_1 + (\Lambda_2 + \Lambda_1)^{(*)} + \Lambda_1^*)$ must be a P.V., which is a contradiction. Thus, together with Lemmas 2.21 and 2.22, we have our result. Q.E.D.

THEOREM 2.24. *All 2-simple P.V.'s of type* II *which contain*

$$\left(GL(1)^4 \times G \times SL(2m+1),\ \rho_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2^* + \Lambda_1 + \Lambda_1\right)\right) \qquad (2 \leqslant \deg \rho_1 \leqslant 2m)$$

as a component are given by

$$(2.20) \qquad \left(GL(1)^4 \times SL(2n+1) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2^* + \Lambda_1 + \Lambda_1\right)\right),$$

$$(2.21) \qquad \left(GL(1)^4 \times SL(2m) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2^* + \Lambda_1 + \Lambda_1\right)\right).$$

PROOF. First assume that $\deg \rho_1 \leqslant 2m-1$. Then it is P.V.-equivalent to $(GL(1)^{3+s} \times G \times Sp(m),\ \rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1))$ by (1) of Proposition 1.34. Since this is of type I, we have (2.20) by [3]. Next assume that $\deg \rho_1 = 2m$. Then we have $(G, \rho_1) = (SL(2m),\ \Lambda_1^{(*)})$ by Lemma 2.11. Clearly (2.21) is castling-equivalent to a regular simple P.V. $(GL(1)^4 \times SL(2m+1),\ \Lambda_2^* + \Lambda_1 + \Lambda_1 + \Lambda_1^*)$. We shall show that

$$\left(GL(1)^{4+s} \times SL(2m) \times SL(2m+1),\ (\sigma_1 + \cdots + \sigma_s) \otimes 1 + \Lambda_1^{(*)} \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2^* + \Lambda_1 + \Lambda_1\right)\right) \qquad (s \geqslant 1, \sigma_1 \neq 1)$$

is a non-P.V. If it is a P.V., by (1) of Proposition 1.34,

$$\left(GL(1)^{3+s} \times SL(2m) \times Sp(m),\ (\sigma_1 + \cdots + \sigma_s)^{(*)} \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1)\right)$$

must be a P.V., which is P.V.-equivalent to $(GL(1)^{2+s} \times Sp(m),\ \sigma_1 + \cdots + \sigma_s + \Lambda_1 + \Lambda_1)$ by Proposition 1.12. Hence we have $s = 1$, $\sigma_1 = \Lambda_1^{(*)}$ by [2]. Since the case $\sigma_1 = \Lambda_1$ and the case $\sigma_1 = \Lambda_1^*$ are reductive P.V.-equivalent (see Remark 1.32), we assume that $\sigma_1 = \Lambda_1$. Then it is castling-equivalent to

$$\left(GL(1)^5 \times SL(2) \times SL(2m+1),\ \Lambda_1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* + 1 \otimes \left(\Lambda_2^* + \Lambda_1 + \Lambda_1\right)\right)$$
$$\cong \Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2 + \Lambda_1^* + \Lambda_1^*\right),$$

which is a non-P.V. by Lemma 2.20. Thus we have a contradiction. Q.E.D.

THEOREM 2.25.

$$\left(GL(1)^{4+s} \times G \times SL(2m+1),\ \rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \left(\Lambda_2 + \Lambda_1^* + \Lambda_1^*\right)\right) \qquad (2 \leqslant \deg \rho_1 \leqslant 2m)$$

is a P.V. if and only if it is

$$(2.22) \quad \left(GL(1)^4 \times SL(2n') \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2 + \Lambda_1^* + \Lambda_1^*\right)\right).$$

PROOF. First note that

$$\left(GL(1)^4 \times SL(m') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2 + \Lambda_1^* + \Lambda_1^*\right)\right) \qquad (2 \leqslant m' \leqslant 2m)$$

is a P.V. if and only if $m' =$ even since it is castling-equivalent to

$$\left(GL(1)^4 \times SL(2m+1-m') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2^* + \Lambda_1 + \Lambda_1\right)\right)$$

(see Theorem 2.24). In particular, $\deg \rho_1$ must be even. Hence by Proposition 1.35 and Theorem 2.19, we have $(G, \rho_1) = (SL(2n'), \Lambda_1)$ and $s = 0$; $s = 1$, $\sigma_1 = \Lambda_1$. However

$$\left(GL(1)^5 \times SL(2n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \left(\Lambda_2 + \Lambda_1^* + \Lambda_1^*\right)\right)$$

is castling-equivalent to

$$\left(GL(1)^5 \times SL(2m+2-2n') \times SL(2m+1),\right.$$
$$\left.\Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \left(\Lambda_2^* + \Lambda_1 + \Lambda_1\right)\right),$$

which is a non-P.V. by Theorem 2.24. Thus $s = 0$ and we obtain our result. Q.E.D.

PROPOSITION 2.26.

$$\left(GL(1)^5 \times G \times SL(2m+1),\ \rho_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_2^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)}\right)\right) \qquad (1 \leqslant n' = \deg \rho_1 \leqslant 2m)$$

is a non-P.V.

PROOF. Without loss of generality, we may assume that $(G, \rho_1) = (SL(n'), \Lambda_1)$. If $n' = 1$ or $n' = 2m$, it is P.V.-equivalent to

$$\left(GL(1)^5 \times SL(2m+1),\ \Lambda_2^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)}\right),$$

which is a non-P.V. by [2]. Therefore, by a castling transformation if necessary, it is enough to consider the $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + 1 \otimes T$ with $1 \lneqq n' \lneqq 2m$, $T = \Lambda_1^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)}$. First assume that $T = \Lambda_1 + \Lambda_1 + \Lambda_1$. By (1) of Proposition 1.34, $(GL(1)^4 \times SL(n') \times Sp(m),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes T)$ must be a P.V., which is a contradiction by Lemma 2.20 and 2.22 in [3]. Next assume that $T = \Lambda_1^* + \Lambda_1^{(*)} + \Lambda_1^{(*)}$. Since the generic isotropy subgroup of $(GL(1)^2 \times SL(2m+1),\ \Lambda_2^* + \Lambda_1^*)$ is $GL(1) \times Sp(m)$ (see p. 95 in [2]), it is isotropic P.V.-equivalent to

$$\left(GL(1)^4 \times SL(n') \times Sp(m),\ \Lambda_1 \otimes (\Lambda_1 + 1) + 1 \otimes (\Lambda_1 + 1) + 1 \otimes (\Lambda_1 + 1)\right).$$

Hence if it is a P.V., then

$$\left(GL(1)^4 \times SL(n') \times Sp(m),\ \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1\right)$$

must be a P.V., which is a contradiction by Lemma 2.23 in [3]. Q.E.D.

THEOREM 2.27.

$$\big(GL(1)^{3+s} \times G \times SL(2m+1),\ \rho_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\Lambda_2 + \Lambda_2)^{(*)} \ \ \big(\textit{or } +1 \otimes \Lambda_1^{(*)} \textit{ when } m = 2\big)\big)$$

$(2 \leqslant \deg \rho_1 \leqslant 2m,\ m \geqslant 2)$ *is a P.V. if and only if it is one of the following.*

(2.23) $\quad \big(GL(1)^3 \times SL(2) \times SL(5),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + 1 \otimes \Lambda_2^*\big).$

(2.24) $\quad \big(GL(1)^3 \times SL(3) \times SL(5),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + 1 \otimes \Lambda_2\big).$

(2.25) $\quad \big(GL(1)^3 \times SL(4) \times SL(5),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + 1 \otimes \Lambda_2\big).$

Here (2.23) *and* (2.24) *are castling-equivalent and they are regular P.V.'s.*

PROOF. First consider the case when $(G, \rho_1) = (SL(n'), \Lambda_1)$. If

$$\big(GL(1)^3 \times SL(n') \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2)^{(*)}\big)$$

is a P.V., then we have $2m \leqslant n' + 2/(n'-1)$ by dimension reasons. Hence, if $n' \geqslant 4$, we have $2m \leqslant n'\ (\leqslant 2m)$, i.e., $n' = 2m$. Thus it is castling-equivalent to $(GL(1)^3 \times SL(2m+1),\ \Lambda_1^* + (\Lambda_2 + \Lambda_2)^{(*)})$ which is a P.V. only when it is $(GL(1)^3 \times SL(5),\ \Lambda_1^* + \Lambda_2 + \Lambda_2)$ by Theorem 1.3 in [3], and we obtain (2.25). If

$$\big(GL(1)^5 \times SL(4) \times SL(5),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2) + \sigma \otimes 1 + 1 \otimes \tau\big)$$

is a P.V., we have $\sigma = \tau = 1$ by dimension reasons. If $n' = 2$ or 3, similarly we have $(2 \leqslant)\ m \leqslant 2$, i.e., $m = 2$. We shall show that $(GL(1)^3 \times SL(2) \times SL(5),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2))$ is a non-P.V. Let

$$\mathfrak{G}_{X_0} = \left\{ \left(\varepsilon_1, \varepsilon_2; \left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_3 \end{array} \right) \right) \right\}$$

be the generic isotropy subalgebra of $(GL(1)^2 \times SL(5),\ \Lambda_2 + \Lambda_2)$ stated in the proof of Proposition 1.1 in [3]. Then it is P.V.-equivalent to

$$(X \mid Y) \mapsto \beta(X \mid Y) + B(X \mid Y) + \big(X{}^tA_1 + Y{}^tA_2 \mid Y{}^tA_3\big)$$

where $B \in \mathfrak{sl}(2)$, $X \in M(2)$, $Y \in M(2,3)$. If it is a P.V., then

$$Y \mapsto \beta Y + BY + Y{}^tA_3 = BY + Y \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \qquad (B \in \mathfrak{sl}(2)),$$

with $\lambda_1 = \beta + \varepsilon_1 + \varepsilon_2$, $\lambda_2 = \beta + 2\varepsilon_2$, $\lambda_3 = \beta + 2\varepsilon_1$, must be a P.V. However, since $\lambda_1, \lambda_2, \lambda_3$ are not independent $(2\lambda_1 = \lambda_2 + \lambda_3)$, it is a non-P.V. Since

$$x \mapsto \beta x + Bx - x \left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_3 \end{array} \right) \qquad (x \in M(2,5))$$

is a P.V. with a generic point

$$x_0 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

$(GL(1)^3 \times SL(2) \times SL(5),\ \Lambda_1 \otimes \Lambda_1^* + 1 \otimes \Lambda_2 + 1 \otimes \Lambda_2)$ is a P.V., i.e., (2.23). By a castling transformation,

$$\big(GL(1)^3 \times SL(3) \times SL(5),$$
$$\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2)\ \big(\text{resp. } \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^* + \Lambda_2^*)\big)\big)$$

is a P.V. (resp. a non-P.V.). Since the generic isotropy subalgebras of (2.23) and (2.24) are $\{0\}$, they are regular and maximal. We say that a P.V. (H, ρ, V) is maximal when $(GL(1) \times H,\ 1 \otimes \rho + \Lambda_1 \otimes \rho')$ is a non-P.V. for any nontrivial irreducible representation ρ' of H. Now assume that $(G, \rho_1) \neq (SL(n),\ \Lambda_1^{(*)})$. Since $\rho_1(G) \subset SL(n')$, it must be of the form

$$\big(GL(1)^3 \times G \times SL(5),\ \rho_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2)^{(*)}\big)$$

with $2 \leqslant \deg \rho_1 \leqslant 4$. Hence (G, ρ_1) must be one of $(SL(2), 2\Lambda_1)$, $(Sp(2), \Lambda_1)$. However, in this case, it is a non-P.V. by dimension reasons. Q.E.D.

3. The case for $(G;\ \rho_1, \dots, \rho_k;\ \sigma_1, \dots, \sigma_s) \neq (SL(m);\ \Lambda_1, \dots, \Lambda_1;\ \Lambda_1^{(*)}, \dots, \Lambda_1^{(*)})$ **with** $(\tau_1, \dots, \tau_t) = (\Lambda_1^{(*)}, \dots, \Lambda_1^{(*)})$. In this section, we shall classify 2-simple P.V.'s of type

$$\big(GL(1)^{k+s+t} \times G \times SL(n),$$
$$(\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \big(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{t}\big)\big)$$

with $2 \leqslant \deg \rho_i \leqslant n\ (i = 1, \dots, k)$ where

$$(G;\ \rho_1, \dots, \rho_k;\ \sigma_1, \dots, \sigma_s) \neq \big(SL(m);\ \Lambda_1, \dots, \Lambda_1;\ \Lambda_1^{(*)}, \dots, \Lambda_1^{(*)}\big).$$

From now on, we shall deal with two cases separately.

3.1. *The case for* $n \lneqq \sum_{i=1}^{k} \deg \rho_i$.

LEMMA 3.1. *Assume that the group* G *is simple and* $2 \leqslant \deg \rho_1 \leqslant n'$ *and* $\deg \rho_2 \gneqq n'$. *Then* $(GL(1)^2 \times G \times SL(n'),\ \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1)$ *is a non-P.V.*

PROOF. Since G is simple, we have $(\deg \rho_1)^2 - 1 \geqslant \dim G$. If it is a P.V., then we have $\dim G + n'^2 + 1 \geqslant (\deg \rho_1 + \deg \rho_2) n' \gneqq (\deg \rho_1) n' + n'^2$, and hence $\deg \rho_1 \gneqq n'$, which is a contradiction. Q.E.D.

PROPOSITION 3.2. *Assume that* $(GL(1)^k \times G \times SL(n),\ (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1)$ *with* $2 \leqslant \deg \rho_i \lneqq n \lneqq \sum_{i=1}^{k} \deg \rho_i\ (i = 1, \dots, k)$ *is a P.V. Put* $n' = \sum_{i=1}^{k} \deg \rho_i - n$. *Then we have* $\deg \rho_i \gneqq n'$ *for* $i = 1, \dots, k$.

PROOF. We shall prove by induction on k. When $k = 2$, it is obvious. Assume that $k \geqslant 3$ and the proposition holds for $k - 1$. If $\deg \rho_i \leqslant n'$ for some i, then we have $\deg \rho_j \lneqq n'$ for all $j = 1, \dots, k$ by Proposition 1.12 and Lemma 3.1. Since $\sum_{i=1}^{k-1} \deg \rho_i \gneqq \sum_{i=1}^{k-1} \deg \rho_i + (\deg \rho_k - n) = n'$,

$$\big(GL(1)^{k-1} \times G \times SL(n'),\ (\rho_1^* + \cdots + \rho_{k-1}^*) \otimes \Lambda_1\big)$$

satisfies the condition of the proposition for $k-1$. By the assumption of the induction, we have $\deg\rho_i \gneq \sum_{j=1}^{k-1}\deg\rho_j - n' = n - \deg\rho_k$ for $i = 1,\ldots,k-1$. Thus we have $\deg\rho_k \gneq n - d$ where $d = \min\{\deg\rho_1,\ldots,\deg\rho_{k-1}\}$. Hence

$$\begin{aligned} k+(d^2-1)+(n^2-1) &\geqslant k+\dim G+n^2-1 \\ &\geqslant (\deg\rho_1+\cdots+\deg\rho_{k-1}+\deg\rho_k)n \\ &\gneq ((k-1)d+(n-d))n \\ &= (k-2)\,dn+n^2, \end{aligned}$$

i.e., $k+d^2-2 \gneq (k-2)\,dn \gneq (k-2)d^2$. Since they are integers, we have $k+d^2-2 \geqslant (k-2)d^2+2$, i.e., $0 \geqslant (k-3)d^2-k+4 \geqslant 3k-8 \geqslant 1$ $(k \geqslant 3)$, which is a contradiction. Q.E.D.

LEMMA 3.3. *Assume that* $k \geqslant 2$, $\deg\rho_i \gneq n' \geqslant 2$ *for* $i = 1,\ldots,k$, *and* $(G;\ \rho_1+\cdots+\rho_k;\ \sigma_1+\cdots+\sigma_s) \neq (SL(m);\ (\Lambda_1+\cdots+\Lambda_1)^{(*)};\ \Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)})$. *Then*

$$\left(GL(1)^{k+s}\times G\times SL(n'), (\rho_1+\cdots+\rho_k)\otimes\Lambda_1+(\sigma_1+\cdots+\sigma_s)\otimes 1\right)$$

is a P.V. if and only if it is

$$\left(GL(1)^2\times SL(4)\times SL(2), (\Lambda_2+\Lambda_1)\otimes\Lambda_1\right).$$

PROOF. If $(G,\rho_1+\cdots+\rho_k) \neq (SL(m), \Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)})$, it is a 2-simple P.V. of type I with $k \geqslant 2$. Hence by §2 in [3], it is $(GL(1)^2\times SL(4)\times SL(2), (\Lambda_2+\Lambda_1)\otimes\Lambda_1)$ (see (2.60) in [3]). If $(G,\rho_1+\cdots+\rho_k) = (SL(m), \Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)})$, it is a non-P.V. except $(G,\rho_1+\cdots+\rho_k) = (SL(m),(\Lambda_1+\cdots+\Lambda_1)^{(*)})$ by Proposition 1.15. In this case, we have $\sigma_1+\cdots+\sigma_s \neq \Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)}$ by our assumption. Such cases are dealt with in §2, and they are non-P.V.'s when $k \geqslant 2$. Q.E.D.

THEOREM 3.4.

$$\left(GL(1)^{k+s}\times G\times SL(n), (\rho_1+\cdots+\rho_k)\otimes\Lambda_1+(\sigma_1+\cdots+\sigma_s)\otimes 1\right)$$
$$\left(2 \leqslant \deg\rho_i \lneq n \lneq \sum_{j=1}^{k}\deg\rho_j \text{ for } i = 1,\ldots,k\right)$$

is a P.V. if and only if it is one of the following.

(3.1) $(GL(1)^2\times SL(4)\times SL(8), (\Lambda_2+\Lambda_1)\otimes\Lambda_1)$.

(3.2) $n = \sum_{i=1}^{k}\deg\rho_i - 1$ *and* $(GL(1)^{k+s}\times G,\ \rho_1^*+\cdots+\rho_k^*+\sigma_1+\cdots+\sigma_s)$ *is a simple P.V.*

PROOF. It is castling-equivalent to

$$\left(GL(1)^{k+s}\times G\times SL(n'), (\rho_1^*+\cdots+\rho_k^*)\otimes\Lambda_1+(\sigma_1+\cdots+\sigma_s)\otimes 1\right)$$

where $n' = \sum_{j=1}^{k}\deg\rho_j - n$. If $n' = 1$, we have the case (3.2). If $n' \geqslant 2$, then by Proposition 3.2, it satisfies the condition of Lemma 3.3, and we have (3.1). Q.E.D.

PROPOSITION 3.5.

$$\big(GL(1)^{k+s+1} \times G \times SL(n),\quad (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_1\big) \quad (k \geqslant 2)$$

is a non-P.V. for $n \lneqq \sum_{j=1}^{k} \deg \rho_j$.

PROOF. In the case of (3.1), $(GL(1)^3 \times SL(4) \times SL(8), (\Lambda_2 + \Lambda_1) \otimes \Lambda_1 + 1 \otimes \Lambda_1)$ is a non-P.V. by dimension reasons. In the case of (3.2), it is castling-equivalent to

$$\big(GL(1)^{k+s+1} \times G \times SL(2),\quad (\rho_1^* + \cdots + \rho_k^*) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_1\big).$$

If $\deg \rho_i = 2$ for some i, it is a non-P.V. by Lemma 1.11 since $k \geqslant 2$. If $\deg \rho_i \gneqq 2$ for $i = 1, \ldots, k$, the possibility is $(GL(1)^3 \times SL(4) \times SL(2), (\Lambda_2 + \Lambda_1) \otimes \Lambda_1 + 1 \otimes \Lambda_1)$ by Lemma 3.3. However it is a non-P.V. by dimension reasons. Q.E.D.

Our next aim is to show that

$$\big(GL(1)^{k+s+1} \times G \times SL(n),\quad (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_1^*\big) \quad \left(k \geqslant 2;\ \sum_{i=1}^{k} \deg \rho_i \gneqq n\right)$$

is a non-P.V.

LEMMA 3.6. *Let G be a subgroup of $GL(n)$ and define a polynomial map ψ: $M(n, n-1) \to V(n) = K^n$ by $\psi(Y) = {}^t(\hat{y}_1, \ldots, \hat{y}_n)$ where $Y = {}^t(y_1, y_2, \ldots, y_n)$ and $\hat{y}_i = (-1)^{i+1}\det(y_1 \cdots \check{y}_i \cdots y_n)$ for $i = 1, \ldots, n$. Then we have $\psi(gY{}^th) = (\det g){}^tg^{-1} \cdot \psi(Y)$ for all $(g, h) \in G \times SL(n-1)$ and $Y \in M(n, n-1)$.*

PROOF. It is enough and easy to check when $g = \alpha I_n$,

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \alpha & & & \\ & & & \ddots & & \\ & & & & \alpha^{-1} & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & & \\ & \ddots\lambda & \\ & & 1 \end{pmatrix}. \quad \text{Q.E.D.}$$

LEMMA 3.7. *For $k \geqslant 2$,*

$$\big(GL(1)^{k+1} \times SL(m) \times SL(km-1), \big(\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}\big) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*\big)$$

is a non-P.V.

PROOF. We identify the space V with $\{(X_1, \ldots, X_k;\ y) \mid X_i \in M(m, km-1)$ for $i = 1, \ldots, k;\ y \in K^{km-1}\}$. Then the action is given by $x \mapsto (\alpha_1 A_1 X_1 {}^tB, \ldots, \alpha_k A_k X_k {}^tB;\ \beta {}^tB^{-1} y)$ for $x = (X_1, \ldots, X_k;\ y) \in V$ and $g = (\alpha_1, \ldots, \alpha_k, \beta;\ A, B) \in GL(1)^{k+1} \times SL(m) \times SL(km-1)$ where A_i stands for A (resp. ${}^tA^{-1}$) when the ith part of $(\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*$ stands for $\Lambda_1 \otimes \Lambda_1$ (resp. $\Lambda_1^* \otimes \Lambda_1$)

for $i = 1, \ldots, k$. Put

$$\begin{pmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_k \end{pmatrix} = \psi \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix}$$

where ψ is the map defined in Lemma 3.6. Then, by Lemma 3.6, we have

$$\hat{X}_1 \mapsto \frac{(\alpha_1 \cdots \alpha_k)^m}{\alpha_1} {}^tA_1^{-1}\hat{X}_1, \qquad \hat{X}_2 \mapsto \frac{(\alpha_1 \cdots \alpha_k)^m}{\alpha_2} {}^tA_2^{-1}\hat{X}_2,$$
$$(X_1 y) \mapsto \alpha_1 \beta A_1(X_1 y) \quad \text{and} \quad (X_2 y) \mapsto \alpha_2 \beta A_2(X_2 y).$$

Hence $F(x) = {}^t(X_2 y)\hat{X}_2/{}^t(X_1 y)\hat{X}_1$ is an absolute invariant. We shall see that $F(x)$ is nonconstant for $k \geqslant 3$. Put $x(y) = (X_1^0, \ldots, X_k^0;\ y)$ where

$$y = {}^t(y_1, \ldots, y_{km-1}) \quad \text{and} \quad \begin{pmatrix} X_1^0 \\ \vdots \\ X_k^0 \end{pmatrix} = \left(\frac{I_{km-1}}{\underbrace{10\cdots 0}_{m}10\cdots 0} \right).$$

Since $k \geqslant 3$, we have $\hat{X}_1 = \hat{X}_2 = {}^t((-1)^{km}, 0, \ldots, 0)$, $X_1 y = {}^t(y_1, \ldots, y_m)$, $X_2 y = {}^t(y_{m+1}, \ldots, y_{2m})$, and we get $F(x(y)) = y_{m+1}y_1^{-1}$. Assume that $k = 2$. For $(\Lambda_1 + \Lambda_1^*) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*$,

$$F(x) = \left\{ {}^t(X_1 y)(X_2 y) \right\} \cdot \left\{ {}^t\hat{X}_1 \hat{X}_2 \right\} \cdot \left\{ {}^t(X_1 y)\hat{X}_1 \right\}^{-2}$$

is an absolute invariant. For $x(y)$ with $k = 2$, we have $\hat{X}_1 = {}^t(1, 0, \ldots, 0)$, $\hat{X}_2 = {}^t(1, 0, \ldots, 0, -1)$, $X_1 y = {}^t(y_1, \ldots, y_m)$ and $X_2 y = {}^t(y_{m+1}, \ldots, y_{2m-1}, y_1 + y_{m+1})$. Hence

$$F(x(y)) = y_1^{-2} \cdot \{ y_1 y_{m+1} + \cdots + y_{m-1}y_{2m-1} + y_1 y_m + y_m y_{m+1} \},$$

i.e., $F(x)$ is a nonconstant absolute invariant. For $(\Lambda_1 + \Lambda_1) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*$ ($\cong (\Lambda_1^* + \Lambda_1^*) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*$),

$$F(x) = \left\{ {}^t(X_1 y)\hat{X}_2 \right\} \cdot \left\{ {}^t(X_2 y)\hat{X}_1 \right\} \cdot \left\{ {}^t(X_1 y)\hat{X}_1 \right\}^{-2}$$

is an absolute invariant. Since $F(x(y)) = (y_1 - y_m)y_{m+1}y_1^{-2}$, $F(x)$ is nonconstant. Thus our space cannot be a P.V. Q.E.D.

PROPOSITION 3.8. *Assume that* $2 \leqslant \deg \rho_i \lneqq n \lneqq \sum_{i=1}^k \deg \rho_i$ *for* $i = 1, \ldots, k$ (*hence* $k \geqslant 2$). *Then*

$$\left(GL(1)^{k+s+1} \times G \times SL(n), (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + \left(\sigma_1 + \cdots + \sigma_s\right) \otimes 1 + 1 \otimes \Lambda_1^* \right)$$

is a non-P.V.

PROOF. Since $(GL(1)^3 \times SL(4) \times SL(8), (\Lambda_2 + \Lambda_1) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$ is a non-P.V. by dimension reasons, if our space is a P.V., $(GL(1)^{k+s} \times G,\ \rho_1 + \cdots + \rho_k + \sigma_1^* + \cdots + \sigma_s^*)$ $(k \geqslant 2)$ must be a simple P.V. and $n = \deg(\rho_1 + \cdots + \rho_k) - 1$ by

Theorem 3.4. By Lemma 3.7, we may assume that $(G, \rho_1 + \cdots + \rho_k) \neq (SL(m), \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$. Since

$$\left(GL(1)^{k+2} \times SL(2m) \times SL(2km-1), \Lambda_2 \otimes 1 + \left(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{k}\right) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*\right)$$

is a non-P.V., we may also assume that $(G, \rho_1 + \cdots + \rho_k) \neq (Sp(m), \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$. Since

$$\begin{aligned}\dim GL(1)^{k+s+1} \times G \times SL(n) &\geqslant \deg\big((\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Lambda_1^*\big)\\ &\geqslant \deg\big((\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*\big) + s\end{aligned}$$

with $n = \deg(\rho_1 + \cdots + \rho_k) - 1$, the simple P.V. $(GL(1)^k \times G, \rho_1 + \cdots + \rho_k)$ must satisfy the condition that $\dim G \geqslant 2(\deg \rho_1 + \cdots + \deg \rho_k - 1) - k$. By Theorem 1.3 in [**3**], such a P.V. must be $(GL(1)^2 \times \mathrm{Spin}(8)$, vector rep. + half-spin rep.). However,

$$\left(GL(1)^3 \times \mathrm{Spin}(8) \times SL(15), (\text{vector rep.} + \text{half-spin rep.}) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*\right)$$

is reductive P.V.-equivalent to (vector rep. + half-spin rep.) $\otimes \Lambda_1 + 1 \otimes \Lambda_1$) (see Remark 1.32), which is a non-P.V. by Proposition 3.5. Q.E.D.

3.2. *The case for* $n \geqslant \sum_{i=1}^k \deg \rho_i$. We shall consider

$$\left(GL(1)^{k+s+t} \times G \times SL(n), (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \left(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{t}\right)\right)$$

when $n \geqslant \sum_{i=1}^k \deg \rho_i$ and

$$(G;\ \rho_1 + \cdots + \rho_k;\ \sigma_1 + \cdots + \sigma_s) \neq \big(SL(m), \Lambda_1 + \cdots + \Lambda_1;\ \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}\big).$$

If $t = 0$, it is a 2-simple P.V. of trivial type (see Definition 1.10), and hence we may assume that $t \geqslant 1$. Since

$$\left(GL(1)^t \times SL(n), \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{t}\right)$$

must be a P.V., we have $t \leqslant n+1$ and $\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}$ must be one of (1) $\Lambda_1^* + \cdots + \Lambda_1^*$, (2) $\Lambda_1^* + \cdots + \Lambda_1^* + \Lambda_1$, (3) $\Lambda_1 + \cdots + \Lambda_1$, (4) $\Lambda_1 + \cdots + \Lambda_1 + \Lambda_1^*$. First assume that $t = n+1$. Put $d = \min\{\deg \rho_1, \ldots, \deg \rho_k\}$. Then we have

$$\begin{aligned}&k + s + (n+1) + (d^2 - 1) + (n^2 - 1)\\ &\qquad\geqslant \dim GL(1)^{k+s+t} \times G \times SL(n)\\ &\qquad\geqslant \deg\left((\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \left(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{n+1}\right)\right)\\ &\qquad\geqslant n \deg(\rho_1 + \cdots + \rho_k) + 2s + n(n+1)\end{aligned}$$

and hence

$$k + d^2 - 1 \geqslant n\ (\deg\rho_1 + \cdots + \deg\rho_k) + s \geqslant n(kd) + s.$$

Thus we have

$$\begin{aligned} 0 &\geqslant (n-kd)kd + (k-1)(kd^2 + d^2 - 1) + s \\ &\geqslant (n-kd)2k + (k-1)(4k+3) + s \qquad (k \geqslant 1). \end{aligned}$$

Since each term is nonnegative, we have $k = 1$, $s = 0$, and $d = n$. Again by dimension reasons, we have $\dim G \geqslant n^2 - 1$, i.e., $G = SL(n)$ and $\rho_1 = \Lambda_1^{(*)}$. It is a 2-simple P.V. of trivial type and also it contradicts by our assumption $(G, \rho_1 + \cdots + \rho_k, \sigma_1 + \cdots + \sigma_s) \neq (SL(m), \Lambda_1 + \cdots + \Lambda_1, \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$. Hence we may assume that $1 \leqslant t \leqslant n$.

THEOREM 3.9. *Assume that* $n \geqslant \sum_{i=1}^k \deg\rho_i$, $(G, \rho_1 + \cdots + \rho_k;\ \sigma_1 + \cdots + \sigma_s) \neq (SL(m), \Lambda_1 + \cdots + \Lambda_1;\ \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$ *and* $1 \leqslant t \leqslant n$. *Then*

$$\Bigl(GL(1)^{k+s+t} \times G \times SL(n),$$

$$(\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Bigl(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t}\Bigr)\Bigr)$$

is a P.V. if and only if

(3.3) $k = 1$ *and*

$$\bigl(GL(1)^{s+t} \times G, \sigma_1 + \cdots + \sigma_s + \overbrace{\rho_1 + \cdots + \rho_1}^{t}\bigr)$$

is a simple P.V. or

(3.4) $t = 1$ *and*

$$\bigl(GL(1)^{k+s} \times G,\ \sigma_1 + \cdots + \sigma_s + \rho_1 + \cdots + \rho_k\bigr)$$

is a simple P.V.

PROOF. By Theorems 1.14 and 1.16, it is P.V.-equivalent to

$$\Bigl(GL(1)^{k+s+t} \times G,\ \sigma_1 + \cdots + \sigma_s + \Bigl(\overbrace{(\rho_1 + \cdots + \rho_k) + \cdots + (\rho_1 + \cdots + \rho_k)}^{t}\Bigr)\Bigr).$$

If $k = 1$ or $t = 1$, the scalar multiplications act independently and it should be a simple P.V. in [2] and Theorem 1.3 in [3]. Assume that $k \geqslant 2$ and $t \geqslant 2$. Then $(GL(1)^{2k} \times G,\ \rho_1 + \rho_1 + \rho_2 + \rho_2 + \cdots + \rho_k + \rho_k)$ must be a simple P.V., and we have $(G, \rho_1 + \cdots + \rho_k) = (SL(m), (\Lambda_1 + \cdots + \Lambda_1)^{(*)})$ and $\sigma_1 + \cdots + \sigma_s = \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}$ by Theorem 1.3 in [3], which is a contradiction by our assumption. Q.E.D.

THEOREM 3.10. *Assume that* $n \geqslant \sum_{i=1}^k \deg\rho_i$, $(G;\ \rho_1 + \cdots + \rho_k;\ \sigma_1 + \cdots + \sigma_s) \neq (SL(m);\ \Lambda_1 + \cdots + \Lambda_1;\ \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$ *and* $t \geqslant 2$. *Then*

$$\Bigl(GL(1)^{k+s+t} \times G \times SL(n),$$

$$(\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Bigl(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t-1} + \Lambda_1\Bigr)\Bigr)$$

is a P.V. if and only if $t = 2, 3$ *and one of the following cases holds.*

(3.5) $\sum_{i=1}^{k}\deg\rho_i + 1 \leqslant n$, $t = 2$, *and* $(GL(1)^{s+k} \times G,\ \sigma_1 + \cdots + \sigma_s + \rho_1 + \cdots + \rho_k)$ *is a simple P.V.*

(3.6) $k = 1$, $\deg\rho_1 + 1 \leqslant n$, $t = 3$, $s = 0,1$; $s = 0$, $(G, \rho_1) = (Sp(m), \Lambda_1)$, $(\mathrm{Spin}(10), \Lambda_e)$, $(SL(2m+1), \Lambda_2)$; $s = 1$, $(G, \sigma_1, \rho_1) = (SL(n'), \Lambda_2, \Lambda_1^*)$, $(SL(2m), \Lambda_2, \Lambda_1)$, $(SL(6), \Lambda_3, \Lambda_1)$.

(3.7) $k = 1$, $\deg\rho_1 = n$, (i) $s = 0$, $t = 2, 3$, $(G, \rho_1) = (Sp(n/2), \Lambda_1)$ (n = even), (ii) $s = 1$, $(G, \rho_1, \sigma_1) = (Sp(n/2), \Lambda_1, \Lambda_1)$ (n = even), $(SL(n), \Lambda_1, \Lambda_2^{(*)})$ ($t = 2$); $(G, \rho_1, \sigma_1) = (SL(n), \Lambda_1, \Lambda_2)$ (n = even), $(SL(n), \Lambda_1, \Lambda_2^*)$ ($t = 3$), (iii) $s = 2$, $t = 2$, $(G, \rho_1, \sigma_1, \sigma_2) = (SL(n), \Lambda_1^{(*)}, \Lambda_2, \Lambda_1)$ (n = even), $(SL(n), \Lambda_1^{(*)}, \Lambda_2, \Lambda_1^*)$.

PROOF. First assume that $\sum_{i=1}^{k}\deg\rho_i + 1 \leqslant n$. Then, by Theorems 1.14 and 1.16, it is P.V.-equivalent to

$$\left(GL(1)^{k+s+t} \times G,\ \sigma_1 + \cdots + \sigma_s + (\rho_1 + \cdots + \rho_k + 1) \otimes \left(\overbrace{1 + \cdots + 1}^{t-1}\right)\right),$$

which is P.V.-equivalent to

$$\left(GL(1)^{k+s} \times G, \sigma_1 + \cdots + \sigma_s + \left(\overbrace{\rho_1 + \cdots + \rho_1}^{t-1}\right) + \cdots + \left(\overbrace{\rho_k + \cdots + \rho_k}^{t-1}\right)\right).$$

If $t \geqslant 4$, it is a non-P.V. by [2]. Even when $\sum_{i=1}^{k}\deg\rho_i = n$ and $t \geqslant 4$, it is a non-P.V. by Theorem 1.14. Hence $t = 2$ or 3. If $t = 2$, the scalar multiplications act independently and we have the case (3.5). If $t = 3$, we have $k = 1$ by the same argument as in Theorem 3.9. Hence it is P.V.-equivalent to $(GL(1)^{s+1} \times G, \sigma_1 + \cdots + \sigma_s + (\rho_1 + \rho_1))$ and we have the case (3.6) by [2]. Next assume that $n = \sum_{i=1}^{k}\deg\rho_i$. Then it is P.V.-equivalent to

$$\begin{aligned}\big(GL(1)^{k+s+t} \times G, \beta_1\sigma_1 + \cdots + \beta_s\sigma_s + \gamma_1(\alpha_1\rho_1 + \cdots + \alpha_k\rho_k)\\ + \gamma_2\left(\alpha_1^{-1}\rho_1^* + \cdots + \alpha_k^{-1}\rho_k^*\right) \{+\gamma_3(\alpha_1\rho_1 + \cdots + \alpha_k\rho_k) \text{ when } t = 3\}\big)\end{aligned}$$

with $(\alpha_1, \ldots, \alpha_k;\ \beta_1, \ldots, \beta_s;\ \gamma_1, \ldots, \gamma_t) \in GL(1)^{k+s+t}$ ($t = 2, 3$). If $k \geqslant 2$, it is a non-P.V. since $\gamma_1\alpha_1\rho_1 + \gamma_2\alpha_1^{-1}\rho_1^* + \gamma_1\alpha_2\rho_2 + \gamma_2\alpha_2^{-1}\rho_2^*$ has a nonconstant absolute invariant. Hence it is P.V.-equivalent to

$$\left(GL(1)^{s+t} \times G, \sigma_1 + \cdots + \sigma_s + \rho_1 + \rho_1^* \{+\rho_1 \text{ when } t = 3\}\right)$$

and we obtain the case (3.7) by [2]. Q.E.D.

THEOREM 3.11. *Assume that* $n \geqslant \sum_{i=1}^{k}\deg\rho_i$ *and* $(G;\ \rho_1 + \cdots + \rho_k;\ \sigma_1 + \cdots + \sigma_s) \neq (SL(m);\ \Lambda_1 + \cdots + \Lambda_1;\ \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$. *Then*

$$\begin{aligned}\Big(GL(1)^{k+s+t} \times G \times SL(n),\\ (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \left(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t}\right)\Big)\end{aligned}$$

is a P.V. if and only if one of the following cases holds.

(3.8) *$n \geqslant t + \sum_{i=1}^{k} \deg \rho_i$ and it is a 2-simple P.V. of trivial type.*

(3.9) *$n = t - 1 + \sum_{i=1}^{k} \deg \rho_i$ and $(GL(1)^{k+s} \times G,\ \sigma_1 + \cdots + \sigma_s + \rho_1^* + \cdots + \rho_k^*)$ is a simple P.V.*

(3.10) *$n = 1 + \sum_{i=1}^{k} \deg \rho_i$, $t \geqslant 3$, and*

$$\Big(GL(1)^{k+s+t} \times G \times SL(n),$$
$$(\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1^* + \cdots + \sigma_s^*) \otimes 1 + 1 \otimes \Big(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t-1} + \Lambda_1\Big)\Big)$$

is a P.V. (see Theorem 3.10).

(3.11) *$n = \sum_{i=1}^{k} \deg \rho_i$; $t \geqslant 2$, $k = 1$ and*

$$\Big(GL(1)^{t+s} \times G,\ \sigma_1^* + \cdots + \sigma_s^* + \overbrace{\rho_1 + \cdots + \rho_1}^{t}\Big)$$

is a simple P.V.

PROOF. Put $r = \sum_{i=1}^{k} \deg \rho_i$. If $n \geqslant r + t$, we have (3.8). Assume that $r \leqslant n \lneqq r + t$. Since

$$\Big(\rho_1 + \cdots + \rho_k + \overbrace{1 + \cdots + 1}^{n-r}\Big) \otimes \Lambda_1$$

is a regular P.V.,

$$\Big(\rho_1 + \cdots + \rho_k + \overbrace{1 + \cdots + 1}^{t}\Big) \otimes \Lambda_1$$

is reductive P.V.-equivalent to

$$(\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + 1 \otimes \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{n-r} + \overbrace{\Lambda_1^* + \cdots +}^{t+r-n} + \Lambda_1^*\Big)$$

and hence

$$\Big(GL(1)^t \times SL(n),\ \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{n-r} + \overbrace{\Lambda_1^* + \cdots +}^{t+r-n} + \Lambda_1^*\Big)$$

must be a P.V. Since $t + r - n \geqslant 1$, we have $n = r$, $n = r + 1$ or $n = t + r - 1$. If $n = t + r - 1$, it is castling-equivalent to $(GL(1)^{k+s} \times G,\ \sigma_1 + \cdots + \sigma_s + \rho_1^* + \cdots + \rho_k^*)$ and we have (3.9). If $n = r + 1$, we may assume $t \geqslant 3$ since otherwise we have (3.8) or (3.9). Since $(\rho_1 + \cdots + \rho_k + 1) \otimes \Lambda_1$ is a regular P.V. by Proposition 1.3, it is P.V.-equivalent to

$$\Big(GL(1)^{k+s+t} \times G \times SL(n),$$
$$(\sigma_1^* + \cdots + \sigma_s^*) \otimes 1 + (\rho_1 + \cdots + \rho_k + 1) \otimes \Lambda_1 + 1 \otimes \Big(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t-1}\Big)\Big)$$

and we have (3.10). If $n = r$, it is reductive P.V.-equivalent to

$$\Big(GL(1)^{k+s+t} \times G \times SL(n),$$
$$(\sigma_1^* + \cdots + \sigma_s^*) \otimes 1 + (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + 1 \otimes \Big(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t}\Big)\Big)$$

and hence we have (3.11) by Theorem 3.9. Q.E.D.

THEOREM 3.12. *Assume that* $n \geqslant \sum_{i=1}^{k} \deg \rho_i$ *and* $(G;\ \rho_1 + \cdots + \rho_k;\ \sigma_1 + \cdots + \sigma_s) \neq \left(SL(m);\ \Lambda_1 + \cdots + \Lambda_1;\ \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}\right)$. *Then*

$$\Big(GL(1)^{k+s+t} \times G \times SL(n), (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-1} + \Lambda_1^*\Big) \qquad (t \geqslant 3)$$

is a P.V. if and only if one of the following cases holds.

(3.12) $n \geqslant t - 1 + \sum_{i=1}^{k} \deg \rho_i$, *and* $(GL(1)^{k+s} \times G,\ \sigma_1 + \cdots + \sigma_s + \rho_1 + \cdots + \rho_k)$ *is a simple P.V.*

(3.13) $k = 1,\ t = 3,\ n = \deg \rho_1,\ s = 0, 1;\ (G, \rho_1) = (Sp(n/2), \Lambda_1)$ $(n = \text{even}, s = 0)$; $(G, \sigma_1, \rho_1) = (SL(n), \Lambda_2, \Lambda_1^*)$ *with* $n = \text{even}$, $(SL(n), \Lambda_2, \Lambda_1)$ $(s = 1)$.

PROOF. Put $r = \sum_{i=1}^{k} \deg \rho_i$. If $n \geqslant r + t - 1$, we have (3.12) by Theorems 1.14 and 1.16. Assume that $r \leqslant n \lneqq r + t - 1$. Since

$$\Big(\rho_1 + \cdots + \rho_k + \overbrace{1 + \cdots + 1}^{n-r}\Big) \otimes \Lambda_1$$

is a regular trivial P.V., it is isotropic P.V.-equivalent to

$$\Big(GL(1)^{k+s+t} \times G,\ \beta_1\sigma_1 + \cdots + \beta_s\sigma_s + \gamma_{n-r+1}\left(\alpha_1^{-1}\rho_1^* + \cdots + \alpha_k^{-1}\rho_k^* + \gamma_1^{-1} + \cdots + \gamma_{n-r}^{-1}\right) + \cdots + \gamma_{t-1}\left(\alpha_1^{-1}\rho_1^* + \cdots + \gamma_{n-r}^{-1}\right) + \gamma_t(\alpha_1\rho_1 + \cdots + \alpha_k\rho_k + \gamma_1 + \cdots + \gamma_{n-r})\Big)$$

where $(\alpha_1, \ldots, \alpha_k;\ \beta_1, \ldots, \beta_s;\ \gamma_1, \ldots, \gamma_t) \in GL(1)^{k+s+t}$. Note that $t - 1 \geqslant n - r + 1$. Since $\gamma_{t-1}(\alpha_1^{-1}\rho_1^* + \alpha_2^{-1}\rho_2^*) + \gamma_t(\alpha_1\rho_1 + \alpha_2\rho_2)$ and $\gamma_{t-1}(\alpha_1^{-1}\rho_1^* + \gamma_1^{-1}) + \gamma_t(\alpha_1\rho_1 + \gamma_1)$ have nonconstant absolute invariants, we have $k = 1$ and $n = r$. Hence it is P.V.-equivalent to

$$\Big(GL(1)^{s+t} \times G,\ \sigma_1 + \cdots + \sigma_s + \overbrace{\rho_1^* + \cdots + \rho_1^*}^{t-1} + \rho_1\Big).$$

By [2] and $(G;\ \rho_1 + \cdots + \rho_k;\ \sigma_1 + \cdots + \sigma_s) \neq \left(SL(m), \Lambda_1 + \cdots + \Lambda_1, \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}\right)$, we have (3.13). Q.E.D.

4. The case for $(G;\ \rho_1, \ldots, \rho_k;\ \sigma_1, \ldots, \sigma_s;\ \tau_1, \ldots, \tau_t) = (SL(m);\ \Lambda_1, \ldots, \Lambda_1;\ \Lambda_1^{(*)}, \ldots, \Lambda_1^{(*)};\ \Lambda_1^{(*)}, \ldots, \Lambda_1^{(*)})$. In this section, we shall classify 2-simple P.V.'s of type

$$\Big(GL(1)^{k+s+t} \times SL(m) \times SL(n), \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\Big) \otimes \Lambda_1 + \Big(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\Big) \otimes 1 + 1 \otimes \Big(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{t}\Big)\Big)$$

with $2 \leqslant m \leqslant n$. Since

$$\left(GL(1)^t \times SL(n),\ \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{t}\right)$$

must be a P.V., we have $t \leqslant n + 1$ and

$$\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{t} = \left(\Lambda_1 + \cdots + \Lambda_1 + \Lambda_1^{(*)}\right)^{(*)}$$

by [2].

4.1. *The case when* $km \leqslant n$. We may assume that $t \geqslant 1$ since otherwise it is a 2-simple P.V. if and only if

$$\left(GL(1)^s \times SL(m),\ \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\right)$$

is a P.V., i.e., $s \leqslant m + 1$ and

$$\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} = \left(\Lambda_1 + \cdots + \Lambda_1 + \Lambda_1^{(*)}\right)^{(*)}.$$

THEOREM 4.1. *Consider*

$$\Bigg(GL(1)^{k+s+t} \times SL(m) \times SL(n),\ \left(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\right) \otimes \Lambda_1$$
$$+ \left(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\right) \otimes 1 + 1 \otimes \left(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-1} + \Lambda_1^*\right)\Bigg) \qquad (t \geqslant 1).$$

(4.1) *When* $n \geqslant km + t - 1$, *it is a P.V. if and only if*

$$\left(GL(1)^{k+s} \times SL(m), \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} + \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\right)$$

is a simple P.V., i.e., (i) $k = 1,\ s \leqslant m$,

$$\left(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\right) = (\Lambda_1 + \cdots + \Lambda_1)^{(*)},$$

(ii) $k \geqslant 2,\ s + k \leqslant m + 1$,

$$\left(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\right) = \Lambda_1 + \cdots + \Lambda_1 + \Lambda_1^{(*)}.$$

(4.2) *When* $km + t - 1 \gneqq n \geqslant km$, *it is a P.V. if and only if* $k = 1$, $m = n$ *and*

$$\left(GL(1)^{s+t} \times SL(m),\ \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} + \overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t-1} + \Lambda_1\right)$$

is a simple P.V., i.e., (i) $t = 1, 2;\ s + t \leqslant m + 1$,

$$\left(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\right) = \Lambda_1 + \cdots + \Lambda_1,$$

(ii) $t \geqslant 1,\ s + t \leqslant m + 1$,

$$\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} = \Lambda_1^* + \cdots + \Lambda_1^*.$$

PROOF. By Theorems 1.14 and 1.16, we have (4.1). When $km + t - 1 \gneq n \gneq km$, by the same argument as the proof of Theorem 3.12, we have $k = 1$ and $m = n$. Then, by Proposition 1.12, we have (4.2). Q.E.D.

THEOREM 4.2. *Consider*

$$\Big(GL(1)^{k+s+t} \times SL(m) \times SL(n), \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\Big) \otimes \Lambda_1$$
$$+ \Big(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\Big) \otimes 1 + 1 \otimes \Big(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t}\Big)\Big)$$
$$(2 \leqslant t \leqslant n + 1,\ km \leqslant n).$$

(4.3) *When $t = n + 1$, it is a P.V. if and only if $k = 1$, $s = 0$, $m = n$.*
(4.4) *When $2 \leqslant t \leqslant n$ and $k \geqslant 2$, it is a P.V. if and only if*

$$\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} = \Lambda_1 + \cdots + \Lambda_1$$

and $s + kt \leqslant m$.
(4.5) *When $2 \leqslant t \leqslant n$ and $k = 1$, it is a P.V. if and only if*

$$\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} = \Lambda_1 + \cdots + \Lambda_1 + \Lambda_1^{(*)}$$

and $s + t \leqslant m + 1$.

PROOF. By the argument in the beginning of §§3.2, we have (4.3). When $2 \leqslant t \leqslant n$, by Theorems 1.14 and 1.16, it is P.V.-equivalent to

$$\Big(GL(1)^{k+s+t} \times SL(m), \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} + \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\Big) \otimes \Big(\overbrace{1 + \cdots + 1}^{t}\Big)\Big).$$

Hence if $k = 1$, we have (4.5). Assume that $k \geqslant 2$. Consider $(GL(1)^5 \times SL(m)$, $(\Lambda_1 + \Lambda_1) \otimes (1 + 1) + \Lambda_1^*, V)$ where $V = \{(x_1, x_2, x_3, x_4;\ y) \mid x_i, y \in k^m\}$. The action is given by $x \mapsto (\alpha\gamma A x_1, \alpha\delta A x_2, \beta\gamma A x_3, \beta\delta A x_4;\ \varepsilon {}^tA^{-1} y)$ for $x = (x_1, x_2, x_3, x_4;$ $y) \in V$ and $g = (\alpha, \beta, \gamma, \delta, \varepsilon;\ A) \in GL(1)^5 \times SL(m)$. Then $F(x) = ({}^tx_1 y)({}^tx_4 y)/({}^tx_2 y)({}^tx_3 y)$ is a nonconstant absolute invariant and hence $\Lambda_1^* + (\Lambda_1 + \Lambda_1) \otimes (1 + 1)$ cannot be a P.V. Therefore

$$\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} \quad \text{must be} \quad \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s}.$$

Since scalar multiplications are not independent,

$$\Big(GL(1)^{k+s+t} \times SL(m), \overbrace{\Lambda_1 + \cdots +}^{s+kt} + \Lambda_1\Big)$$

is a P.V. if and only if $s + kt \leqslant m$ by [2]. Q.E.D.

THEOREM 4.3. *Consider*

$$\Big(GL(1)^{k+s+t} \times SL(m) \times SL(n), \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\Big) \otimes \Lambda_1$$
$$+ \Big(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\Big) \otimes 1 + 1 \otimes \Big(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t-1} + \Lambda_1\Big)\Big) \qquad (t \geqslant 3).$$

(4.6) *When $n \geqslant km$, it is a P.V. if and only if*

$$\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} = \Lambda_1 + \cdots + \Lambda_1,$$

$s + k(t-1) \leqslant m$ *(and $k = 1$ when $n = km$).*

PROOF. First assume that $n \geqslant km + 1$. By Theorems 1.14 and 1.16, it is P.V.-equivalent to

$$\Big(GL(1)^{k+s+t} \times SL(m), \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}$$
$$+ \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} + 1\Big) \otimes \Big(\overbrace{1 + \cdots + 1}^{t-1}\Big)\Big),$$

which is also P.V.-equivalent to

$$\big(GL(1)^{k+s} \times SL(m), \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}$$
$$+ (\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-1}) \overbrace{+ \cdots +}^{k} (\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-1})\big).$$

Clearly

$$\big(GL(1) \times GL(1) \times SL(m), \Lambda_1 \otimes 1 \otimes \Lambda_1^* + 1 \otimes \Lambda_1 \otimes (\Lambda_1 + \Lambda_1)\big)$$

is a non-P.V., and hence

$$\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} = \Lambda_1 + \cdots + \Lambda_1.$$

Since scalar multiplications are not independent,

$$\big(GL(1)^{k+s} \times SL(m), \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s+k(t-1)}\big)$$

is a P.V. if and only if $s + k(t-1) \leqslant m$ by [**2**]. When $n = km$, we have $k = 1$ and $n = m$ by the similar argument as the proof of Theorem 3.10. Then, by Proposition 1.12, it is P.V.-equivalent to

$$\Big(GL(1)^{s+t} \times SL(m), \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} + \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-1} + \Lambda_1^*\Big) \qquad (t \geqslant 3)$$

and we obtain our assertion. Q.E.D.

THEOREM 4.4. *Consider*

$$\Big(GL(1)^{k+s+t} \times SL(m) \times SL(n), \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\big) \otimes \Lambda_1$$
$$+ \big(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\big) \otimes 1 + 1 \otimes \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t}\big)\Big).$$

(4.7) *If $n \geqslant km + t$, it is a P.V. if and only if*

$$\Big(GL(1)^{s} \times SL(m), \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\Big)$$

is a simple P.V.

(4.8) *When $km + t \gneq n \gneq km + 1$, it is a P.V. if and only if $n = km + t - 1$ and*

$$\Big(GL(1)^{k+s} \times SL(m), \overbrace{\Lambda_1^{*} + \cdots + \Lambda_1^{*}}^{k} + \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\Big)$$

is a simple P.V.

(4.9) *When $n = km + 1$, it is P.V.-equivalent to*

$$\big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\big) \otimes \Lambda_1 + \big(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\big)^{*} \otimes 1 + 1 \otimes \big(\overbrace{\Lambda_1^{*} + \cdots + \Lambda_1^{*}}^{t-1} + \Lambda_1\big)$$

(*see Theorem* 4.3).

(4.10) *When $n = km$, it is P.V.-equivalent to*

$$\big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\big) \otimes \Lambda_1 + \big(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}\big)^{*} \otimes 1 + 1 \otimes \big(\overbrace{\Lambda_1^{*} + \cdots + \Lambda_1^{*}}^{t}\big)$$

(*see Theorem* 4.2).

PROOF. If $n \geqslant km + t$, it is of trivial type (see Definition 1.1) and we have (4.7). Assume that $km \lneq n \lneq km + t$. By the same argument as the proof of Theorem 3.11, we have $n = km + t - 1$, $km + 1$, or km. When $n = km + t - 1$, we have (4.8) by a castling transformation. If $n = km + 1$ (resp. $n = km$),

$$\big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} + 1\big) \otimes \Lambda_1 \quad \Big(\text{resp. } \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\big) \otimes \Lambda_1\Big)$$

is a regular trivial P.V., we have (4.9) (resp. (4.10)) by Remark 1.32. Q.E.D.

4.2. *The case when $km \gneq n$.*

THEOREM 4.5. *Assume that $k \geqslant 2$. Then*

$$\Big(GL(1)^{k} \times SL(m) \times SL(n), \Lambda_1 \otimes \overbrace{\Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\Big) \qquad (m, n \geqslant 2)$$

is a P.V. if and only if $m \neq n$ and $\dim G \geqslant \dim V$, *i.e., $k + (m^2 - 1) + (n^2 - 1) \geqslant kmn$. More precisely, we have*

(4.11) *If* $\dim G \gneq \dim V$ *(hence $m \neq n$), it is castling-equivalent to a trivial P.V.*

(4.12) *If* $\dim G = \dim V$ *and $m \neq n$, it is castling-equivalent to*

$$\Big(GL(1)^{k} \times SL(k-1), \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\Big).$$

PROOF. If it is a P.V., then $m \neq n$ by Proposition 1.12. Now assume that $m \neq n$ and $\dim G \geqslant \dim V$. If $n \geqslant km$, it is a trivial P.V. If $km \gneqq n \gneqq m$, then it is castling-equivalent to

$$\left(GL(1)^k \times SL(m) \times SL(n'),\ \Lambda_1 \otimes \Lambda_1 \overbrace{+ \cdots +}^{k} \Lambda_1 \otimes \Lambda_1\right)$$

with $n' = km - n$. Since $\dim G \geqslant \dim V$, we have $k \leqslant (m^2 + n^2 - 2)/mn - 1 \lneqq n/m + 1$, i.e., $n' \lneqq m$. If $m \geqslant kn'$, it is a trivial P.V. If $kn' \gneqq m$, then we have $m' = kn' - m \lneqq n'$ and we can go ahead similarly. By repeating this procedure, it can be reduced to a trivial P.V. or a simple P.V.

$$\left(GL(1)^k \times SL(m_0),\ \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\right) \quad \text{with } k < m_0 + 1.$$

If $k \leqslant m_0$, it is a trivial P.V. and we have our result. Note that $\dim G = \dim V$ if and only if the generic isotropy subalgebra is $\{0\}$. Q.E.D.

According to (4.11), we shall consider conversely, namely, we start from a trivial P.V.

LEMMA 4.6. *Let (P_0) be a trivial P.V.*

$$\left(GL(1)^k \times SL(m) \times SL(n),\ \Lambda_1 \otimes \Lambda_1 \overbrace{+ \cdots +}^{k} \Lambda_1 \otimes \Lambda_1\right)$$

with $k \geqslant 2$ and $km \leqslant n$. Then, by the j-times castling transformation, we obtain a P.V. (P_j), as follows.

(4.13) (P_j):

$$\left(GL(1)^k \times SL(a_j n - a_{j-1} m) \times SL(a_{j+1} n - a_j m),\ \Lambda_1 \otimes \Lambda_1 \overbrace{+ \cdots +}^{k} \Lambda_1 \otimes \Lambda_1\right)$$

with $a_j n - a_{j-1} m \lneqq a_{j+1} n - a_j m$ where $\{a_j\}$ is given by $a_{-1} = -1$, $a_0 = 0$ and $a_j = k a_{j-1} - a_{j-2}$ $(j \geqslant 1)$, namely, $a_j = j$ $(k = 2)$, and

$$a_j = \frac{1}{\sqrt{k^2 - 4}} \left\{ \left(\frac{k + \sqrt{k^2 - 4}}{2} \right)^j - \left(\frac{k - \sqrt{k^2 - 4}}{2} \right)^j \right\} \qquad (k \geqslant 3).$$

PROOF. We shall prove by induction on j. If $j = 0$, we have $a_0 n - a_{-1} m = m$ and $a_1 n - a_0 m = n$, i.e., the lemma holds. Assume that the lemma holds for $j - 1$. Then we have (P_{j-1}):

$$\left(GL(1)^k \times SL(a_{j-1} n - a_{j-2} m) \times SL(a_j n - a_{j-1} m),\ \Lambda_1 \otimes \left(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\right)\right)$$

with $a_{j-1} n - a_{j-2} m \lneqq a_j n - a_{j-1} m$. By a castling transformation, we have

$$\left(GL(1)^k \times SL(n') \times SL(a_j n - a_{j-1} m), \Lambda_1 \otimes \Lambda_1 \overbrace{+ \cdots +}^{k} \Lambda_1 \otimes \Lambda_1\right)$$

where

$$\begin{aligned} n' &= k(a_j n - a_{j-1} m) - (a_{j-1} n - a_{j-2} m) \\ &= (k a_j - a_{j-1}) n - (k a_{j-1} - a_{j-2}) m = a_{j+1} n - a_j m, \end{aligned}$$

i.e., (P_j). Note that we have $n' \gneq a_j n - a_{j-1}m$ by the proof of Theorem 4.5, or by direct calculation. Q.E.D.

From now to Lemma 4.11, we use $GL(1)^{k+s+t} \times GL(m) \times GL(n)$ instead of $GL(1)^{k+s+t} \times SL(m) \times SL(n)$ for simplicity. This causes no essential change.

LEMMA 4.7. *Let*

$$\left(GL(1)^k \times GL(m) \times GL(n),\ \Lambda_1 \otimes \Lambda_1 \overbrace{+ \cdots +}^{k} \Lambda_1 \otimes \Lambda_1, V\right)$$

be a triplet with

$$V = M(m,n) \overbrace{\oplus \cdots \oplus}^{k} M(m,n)$$

and $m \lneq n$. *Then the isotopy subalgebra* $\mathfrak{G}_x$ *at* $x = ([I_m|0],\ X_1, \ldots, X_{k-1}) \in V$ *is given by*

$$(4.14)\qquad \mathfrak{G}_x = \left\{\left(\beta_k, \beta_1, \ldots, \beta_{k-1};\ -{}^tB_1 - \beta_k I_m;\ B = \begin{bmatrix} B_1 & B_{12} \\ 0 & B_2 \end{bmatrix}\right) \in \mathfrak{gl}(1)^k \oplus \mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \mid (\beta_i - \beta_k)X_i - {}^tB_1 X_i + X_i {}^tB = 0 \text{ for } i = 1, \ldots, k-1\right\}.$$

PROOF. For

$$\tilde{A} = \left(\beta_k, \beta_1, \ldots, \beta_{k-1};\ A, B = \begin{bmatrix} B_1 & B_{12} \\ B_{21} & B_2 \end{bmatrix}\right) \in \mathfrak{G} = \mathrm{Lie}(G) = \mathfrak{gl}(1)^k \oplus \mathfrak{gl}(m) \oplus \mathfrak{gl}(n),$$

we have

$$d\rho(\tilde{A})x = \{[\beta_k I_m + A + {}^tB_1 | {}^tB_{21}],\ \beta_1 X_1 + AX_1 + X_1 {}^tB, \ldots, \beta_{k-1}X_{k-1} + AX_{k-1} + X_{k-1}{}^tB\}.$$

By definition, $\mathfrak{G}_x = \{\tilde{A} \in \mathfrak{G};\ d\rho(\tilde{A})x = 0\}$, and we have our result. Q.E.D.

LEMMA 4.8. *Let*

$$\left(GL(1)^k \times GL(n) \times GL(kn-m),\ \Lambda_1 \otimes \Lambda_1 \overbrace{+ \cdots +}^{k} \Lambda_1 \otimes \Lambda_1, V'\right)$$

be a castling transform of the triplet in Lemma 4.7. *We identify* V' *with*

$$M(n, kn-m) \overbrace{\oplus \cdots \oplus}^{k} M(n, kn-m)$$

and take a point $x' = ([I_n|0]\,(= X_0'),\ X_1', \ldots, X_{k-1}')$ *of* V' *where*

$$\begin{pmatrix} X_0' \\ \vdots \\ X_{k-2}' \end{pmatrix} = (I_{(k-1)n}|0)$$

and

$$X_{k-1}' = \left(\begin{array}{c|c} X_1, \ldots, X_{k-1} & 0 \\ \hline 0 & I_{n-m} \end{array}\right).$$

Then the isotropy subalgebra $\mathfrak{G}'_{x'}$ *at* x' *is given by*

$$(4.15)\quad \mathfrak{G}'_{x'} = \left\{ \left(-\beta_1, \dots, -\beta_k; -{}^tB\left(= -{}^t\begin{bmatrix} B_1 & B_{12} \\ 0 & B_2 \end{bmatrix}\right); \right.\right.$$

$$\left.\left.\left(\begin{array}{ccc|c} \beta_1 I_n + B & & & \begin{pmatrix} {}^tX_1 \\ \vdots \\ {}^tX_{k-1} \end{pmatrix} B_{12} \\ & \ddots & & \\ & & \beta_{k-1}I_n + B & \\ \hline & 0 & & \beta_k I_{n-m} + B_2 \end{array}\right)\right) \right.$$

$$\in \mathfrak{gl}(1)^k \oplus \mathfrak{gl}(n) \oplus \mathfrak{gl}(kn-m) \mid$$

$$\left. (\beta_i - \beta_k)X_i - {}^tB_1X_i + X_i{}^tB = 0 \textit{ for } i = 1, \dots, k-1 \right\}.$$

In particular, we have $\dim \mathfrak{G}'_{x'} = \dim \mathfrak{G}_x$.

PROOF. For

$$\tilde{A}' = \left(\beta'_1, \dots, \beta'_k;\ B' = \begin{bmatrix} B'_1 & B'_{12} \\ B'_{21} & B'_2 \end{bmatrix};\ C' = \begin{bmatrix} C'_1 & C'_{12} \\ C'_{21} & C'_2 \end{bmatrix}\right) \in \mathfrak{G}'$$

with $B'_1 \in M(m)$ and $C'_1 \in M((k-1)n)$, we have

$$d\rho(\tilde{A}')x' = \begin{bmatrix} \beta'_1 I_n + B' & & \\ & \ddots & \\ & & \beta'_k I_n + B' \end{bmatrix} \left[\begin{array}{cc} I_{(k-1)n} & 0 \\ \hline \multicolumn{2}{c}{X'_{k-1}} \end{array}\right]$$

$$+ \left[\begin{array}{cc} I_{(k-1)n} & 0 \\ \hline \multicolumn{2}{c}{X'_{k-1}} \end{array}\right] \begin{bmatrix} {}^tC'_1 & {}^tC'_{21} \\ {}^tC'_{12} & {}^tC'_2 \end{bmatrix}$$

$$= \left(\begin{array}{c|c} \begin{pmatrix} \beta'_1 I_n + B' & & \\ & \ddots & \\ & & \beta'_{k-1} I_n + B' \end{pmatrix} + {}^tC'_1 & {}^tC'_{21} \\ \hline \multicolumn{2}{c}{\beta'_k X'_{k-1} + B'X'_{k-1} + X'_{k-1}{}^tC'} \end{array}\right) = 0$$

and hence

$$C'_1 = -{}^t\begin{pmatrix} \beta'_1 I_n + B' & & \\ & \ddots & \\ & & \beta'_{k-1} I_n + B' \end{pmatrix}$$

and $C'_{21} = 0$. Therefore, we have

$$\beta'_k X'_{k-1} + B' X'_{k-1} + X'_{k-1}{}^t C' = \left(\begin{array}{c|c} F & B'_{12} \\ \hline {}^t C'_{12} + B'_{21}(X_1, \ldots, X_{k-1}) & {}^t C'_2 + \beta'_k I_{n-m} + B'_2 \end{array}\right)$$
$$= 0$$

where

$F = \left(\left(\beta'_k - \beta'_1\right) X_1 + B'_1 X_1 - X_1 B', \ldots, \left(\beta'_k - \beta'_{k-1}\right) X_{k-1} + B'_1 X_{k-1} - X_{k-1} B'\right)$.

Thus, by putting $\beta_i = -\beta'_i$ $(i = 1, \ldots, k)$ and

$$B = -{}^t B' = \begin{bmatrix} B_1 & B_{12} \\ B_{21} & B_2 \end{bmatrix},$$

we obtain our result. Q.E.D.

Let

$$P_0 = \Big(GL(1)^k \times GL(m) \times GL(n),$$
$$\Lambda_1 \otimes \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} \otimes \Lambda_1,\ M(m,n) \oplus \cdots \oplus M(m,n)\Big)$$

be a trivial P.V. with $km \leqslant n$. Take a generic point $x_0 = ([I_m|0]\ (= X_0^0), X_1^0, \ldots, X_{k-1}^0)$ such that

$$\begin{pmatrix} X_0^0 \\ \vdots \\ X_{k-1}^0 \end{pmatrix} = (I_{km} \quad 0).$$

We define inductively the point $x_j = ([I_r\ 0]\ (= X_0^j), X_1^j, \ldots, X_{k-1}^j)$ of

$$P_j = \Big(GL(1)^k \times GL(a_j n - a_{j-1} m) \times GL(a_{j+1} n - a_j m),$$
$$\Lambda_1 \otimes \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} \otimes \Lambda_1\Big)$$

by

$$\begin{pmatrix} X_0^j \\ \vdots \\ X_{k-2}^j \end{pmatrix} = \left(I_{(k-1)r} \quad 0\right) \quad \text{and} \quad X_{k-1}^j = \left(\begin{array}{c|c} X_1^{j-1} \ldots X_{k-1}^{j-1} & 0 \\ \hline 0 & I \end{array}\right)$$

with $r = a_j n - a_{j-1} m$. Then, by Lemmas 4.7 and 4.8, the point x_j is a generic point of a P.V. P_j and the isotropy subalgebra $\mathfrak{G}_{x_j}$ at x_j is of the form $\mathfrak{G}_{x_j} = \{(\alpha_1, \ldots, \alpha_k; -{}^t A'_{j-1}, A'_j)\}$. First we shall consider

$$P'_j = \Big(GL(a_j n - a_{j-1} m) \times GL(a_{j+1} n - a_j m),\ \Lambda_1 \otimes \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} \otimes \Lambda_1\Big)$$

instead of P_j. Then the isotropy subalgebra $\mathfrak{G}'_{x_j}$ at x_j is of the form $\mathfrak{G}'_{x_j} = \{(-{}^tA_{j-1}, A_j)\}$. For simplicity, we denote

$$\begin{pmatrix} A' & & \\ & \ddots \overset{r}{} & \\ & & A' \end{pmatrix}$$

by A'^r for any matrix A'.

LEMMA 4.9. *Put $-{}^tA_{-1} = A \in \mathfrak{gl}(m)$. Then we have*

$$(4.16)\quad -{}^tA_0 = \left(\begin{array}{c|c|c} A & & \\ \hline & A^{k-1} & \\ \hline F_1 & F_2 & F \end{array}\right), \qquad -{}^tA_1 = \left(\begin{array}{c|c|c} -{}^tA_0^{k-1} & & \\ \hline & A^{k-1} & \\ \hline * & F_2 & F \end{array}\right),$$

$$-{}^tA_j = \begin{array}{r} \\ \\ \\ \\ \\ n-km\{ \end{array}\left(\begin{array}{ccccc|c|c} -{}^tA_{j-1}^{k-1} & & & & & & \\ & -{}^tA_{j-2}^{k-2} & & & & & \\ & & \ddots & & & & \\ & & & -{}^tA_0^{k-2} & & & \\ & & & & & A^{k-1} & \\ \hline & & * & & & F_2 & F \end{array}\right) \qquad (j \geqslant 2).$$

Here $(*)$ is independent of F_2. The number of A contained in $-{}^tA_j$ is a_{j+2}.

PROOF. The former part is inductively obtained from (4.14) and (4.15) (putting $\beta_1 = \cdots = \beta_k = 0$). Note that if

$$B_{12} = \Big(0 \,|\, \overbrace{\quad * \quad}^{n-km}\Big),$$

in (4.15), which is independent of F_2, then

$$\begin{pmatrix} {}^tX_1 \\ \vdots \\ {}^tX_{k-1} \end{pmatrix} B_{12}$$

is also of the form

$$\Big(0 \,|\, \overbrace{\quad * \quad}^{n-km}\Big),$$

independent of F_2. Let b_j be the number of A contained in $-{}^tA_j$. Then $b_0 = k = a_2$. Assume that $b_{j-1} = a_{j+1}$. By (4.16), we have

$$b_j = (k-1)b_{j-1} + \left(b_{j-1} - b_{j-2}\right) = ka_{j+1} - a_j = a_{j+2}. \quad \text{Q.E.D.}$$

LEMMA 4.10. *For $j \geqslant 0$,*

$$\Big(GL(a_jn - a_{j-1}m) \times GL(a_{j+1}n - a_jm), \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s}\Big) \otimes 1 + \Big(\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\Big) + 1 \otimes \Big(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t}\Big)\Big)$$

is a P.V. if and only if $sa_{j+1} + ta_{j+2} \leqslant m$.

PROOF. The generic isotropy subalgebra of

$$P_j' = \Big(GL(a_jn - a_{j-1}m) \times GL(a_{j+1}n - a_jm), \overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\Big)$$

is $\{(-{}^tA_{j-1}, A_j)\}$ where $-{}^tA_j$ is of the form (4.16). Hence, if our space is a P.V.,

$$\Big(GL(m), \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{r}\Big) \quad \text{with } r = sa_{j+1} + ta_{j+2}$$

must be a P.V., and we obtain $sa_{j+1} + ta_{j+2} \leqslant m$. However, thanks to F_2 in (4.16), the converse is also true. Q.E.D.

LEMMA 4.11. *For $j \geqslant 1$ and $t \geqslant 1$,*

$$\Big(GL(1)^{k+s+t} \times GL(a_jn - a_{j-1}m) \times GL(a_{j+1}n - a_jm), \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s}\Big) \otimes 1 + \Big(\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\Big) + 1 \otimes \Big(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t}\Big)\Big)$$

is a P.V. if and only if $sa_{j+1} + ta_{j+2} \leqslant m$.

PROOF. Assume that it is a P.V. The generic isotropy subalgebra of

$$P_j = \Big(GL(1)^k \times GL(a_jn - a_{j-1}m) \times GL(a_{j+1}n - a_jm), \overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\Big)$$

is given by $\mathfrak{G}_{x_j} = \{(\alpha_1, \ldots, \alpha_k; -{}^tA'_{j-1}, A'_j)\}$ where $-{}^tA'_j$ is obtained from $-{}^tA_j$ in (4.16) by replacing A by $\gamma_1 I + A, \ldots, \gamma_r I + A$ $(r = a_{j+2})$ for some scalar multiplications $\gamma_1, \ldots, \gamma_r$. Hence we have $sa_{j+1} + ta_{j+2} \leqslant m + 1$ and the equality holds only when scalar multiplications act independently. By Lemmas 4.7 and 4.8, we have

$$\{\gamma_1, \ldots, \gamma_r\} = \{\alpha_i - \alpha_{i'}\}_{(i \neq i')} \cup \Big\{\overbrace{0, \ldots, 0}^{k-1}\Big\} \quad \text{for } j = 1,\ r = a_3 = k^2 - 1.$$

Since $t \geqslant 1$ and $\gamma_1' = \alpha_1 - \alpha_k + \lambda$, $\gamma_2' = \lambda$, $\gamma_3' = \alpha_k - \alpha_1 + \lambda$ are not independent for any λ (i.e. $\gamma_1' + \gamma_3' = 2\gamma_2'$), scalar multiplications cannot be independent and we have $sa_{j+1} + ta_{j+2} \leqslant m$. The converse is obvious by Lemma 4.10. Q.E.D.

DEFINITION 4.12. Let R be the set of triplets (k, m, n) of natural numbers satisfying $k \geqslant 2$, $n \gneqq m \geqslant 2$ and $k + m^2 + n^2 \gneqq kmn + 2$. For $(k, m, n) \in R$, there exists a nonnegative integer j such that

$$\left(GL(1)^k \times SL(m) \times SL(n),\ \overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\right)$$

is transformed to a trivial P.V. by j-times castling transformations (see (4.11)). This number j is uniquely determined if we use only castling transformations which make the representation degree smaller. We denote such j by $\psi(k, m, n)$. Thus we obtain a map ψ: $R \to Z_+ = \{0, 1, 2, \dots\}$. For example, $\psi(k, m, n) = 0$ if and only if $mk \leqslant n$. Note that $\psi(k, m, n) = \min\{i;\ C_i \leqslant 0\}$ where $C_{-2} = n$, $C_{-1} = m$, $C_i = kC_{i-1} - C_{i-2}\,(i \geqslant 0)$.

THEOREM 4.13. *Consider*

$$\left(GL(1)^{k+s+t} \times SL(m) \times SL(n),\ \left(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s}\right) \otimes 1 + \left(\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\right) + 1 \otimes \left(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t}\right)\right)$$

with $km \gneqq n \gneqq m \geqslant 2$ *and* $t \geqslant 1$. *This is a P.V. if and only if* $(k, m, n) \in R$ *and*

$$sa_{j+1} + ta_{j+2} \leqslant a_{j+1}m - a_j n \quad \left(\Leftrightarrow s + kt \leqslant m - (a_j/a_{j+1})(n - t)\right)$$

for $j = \psi(k, m, n)$ *where* ψ: $R \to Z_+$ *is the map defined in Definition* 4.12, *and* $\{a_i\}$ *is defined by* $a_{-1} = -1$, $a_0 = 0$, $a_i = ka_{i-1} - a_{i-2}$ $(i \geqslant 1)$.

PROOF. Put $m = a_j n' - a_{j-1} m'$ and $n = a_{j+1} n' - a_j m'$. Then we have $m' = a_{j+1} m - a_j n$, and hence we obtain our result from Lemma 4.11. Note that $a_j^2 - a_{j+1} a_{j-1} = 1$, which is obtained by taking the determinant of

$$\begin{pmatrix} a_j & a_{j+1} \\ a_{j-1} & a_j \end{pmatrix} = \begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{j-1} & a_j \\ a_{j-2} & a_{j-1} \end{pmatrix}. \quad \text{Q.E.D.}$$

LEMMA 4.14. *Assume that*

$$\left(GL(1)^{k+s+t} \times SL(m) \times SL(n),\ \left(\overbrace{\Lambda_1 + \cdots + \Lambda}^{s_1} + \overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{s_2}\right) \otimes 1 + \left(\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\right) + 1 \otimes \left(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t_1} + \overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t_2}\right)\right)$$

$$(s \geqslant 1 \text{ or } t \geqslant 1)$$

with $km \gneqq n \gneqq m \geqslant 2$, $s = s_1 + s_2$, $t = t_1 + t_2$, *is a P.V. Then one of the following cases holds.*

(I-a) $s_2 = t_1 = 0$, $t_2 \geqslant 1$,
(I-b) $s_2 = t_2 = 0$,
(I-c) $s_1 = t_2 = 0$, $s_2 \geqslant 1$.
(II-a) $s_2 = 0$, $t_1 \geqslant 1$, $t_2 = 1$,
(II-b) $s_1 = 1$, $s_2 \geqslant 1$, $t_2 = 0$.
(III-a) $s_2 = 0$, $t_1 = 1$, $t_2 \geqslant 2$,
(III-b) $s_1 \geqslant 2$, $s_2 = 1$, $t_2 = 0$.
(IV) $s_2 = t_2 = 1$.

PROOF. Since $\Lambda_1 + \cdots + \Lambda_1 + \Lambda_1^* + \cdots + \Lambda_1^*$ must be $(\Lambda_1 + \cdots + \Lambda_1 + \Lambda_1^{(*)})^{(*)}$ by [2], it is enough to show that if (a) $s_2 \geqslant 1$, $t_2 \geqslant 2$, or (b) $s_2 \geqslant 2$, $t_2 \geqslant 1$, it is a non-P.V. By Theorem 1.14, the prehomogeneity of

$$\left(GL(1)^5 \times SL(m') \times SL(n'),\ \Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes \left(\Lambda_1^* + \Lambda_1^*\right)\right)$$

implies that of $(GL(1)^4 \times SL(n'),\ \Lambda_1 + \Lambda_1 + \Lambda_1^* + \Lambda_1^*)$ for any $m',\ n' \geqslant 2$, which is a contradiction. Thus we obtain our result. Q.E.D.

THEOREM 4.15. *The cases* (I-b) *and* (I-c) *in Lemma* 4.14 *are castling-equivalent to the case dealt with in* §4.1 *or to the case* (I-a) *which is already classified in Theorem* 4.13.

PROOF. The case (I-b) is castling-equivalent to

$$\left(GL(1)^{k+st} \times SL(n') \times SL(m),\ \left(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t_1}\right) \otimes 1 \right.$$
$$\left. + \left(\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\right) + 1 \otimes \left(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{s_1}\right)\right)$$

with $n' = km + t_1 - n$. If it is a P.V., we have $n' \lneqq m$, i.e., $t_1 \lneqq n - (k-1)m$. In fact,

$$\left(GL(1)^{k+n-(k-1)m} \times SL(m) \times SL(n),\right.$$
$$\left. \overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k} + 1 \otimes \left(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{n-(k-1)m}\right)\right)$$

is castling equivalent to

$$\left(GL(1)^{k+n-(k-1)m} \times SL(m) \times SL(m),\right.$$
$$\left. \left(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} + 1 + \cdots + 1\right) \otimes \Lambda_1\right) \qquad (k \geqslant 2),$$

which is a non-P.V. by Proposition 1.12. If $kn' \leqslant m$, it is the case dealt with in §4.1. If $kn' \gneqq m$ and $s_1 \geqslant 1$, it is the case (I-a). If $kn' \gneqq m$ and $s_1 = 0$, then $t_1 \geqslant 1$ and it is castling-equivalent to

$$\left(GL(1)^{k+t_1} \times SL(m') \times SL(n'),\right.$$
$$\left. \overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k} + 1 \otimes \left(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t_1}\right)\right)$$

with $m' = kn' - m \lneqq n'$. If $km' \leqslant n'$, it is the case in §4.1, and if $km' \gneqq n'$, it is the case (I-a). Similarly one can show that the case (I-c) reduces to (I-b) or to the case in §4.1. Q.E.D.

THEOREM 4.16. (1) *The case* (II-b) *in Lemma* 4.14 *is transformed to the case* (II-a) *or to the case in* §4.1 *by a castling transformation.* (2) *Consider the case* (II-a), i.e.,

$$\Big(GL(1)^{k+s+t} \times SL(m) \times SL(n), \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s}\big) \otimes 1 + \big(\Lambda_1 \otimes \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} \otimes \Lambda_1\big) + 1 \otimes \big(\Lambda_1^* + \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-1}\big)\Big)$$

$$(t \geqslant 2, km \gneq n \gneq m).$$

Put $n' = km + t - n - 1$. *Then it is a P.V. if and only if* (i) $kn' \leqslant m$ *and* $(k+s)k + t - 2 \leqslant n'$ *or* (ii) $kn' \gneq m$, $(k, n', m) \in R$ *and* $(t-2)a_{j+1} + (k+s)a_{j+2} \leqslant a_{j+1}n' - a_j m$ *with* $j = \psi(k, n', m)$ (*see Definition* 4.12).

PROOF. (1) By a castling transformation, (II-b) is transformed to

$$\Big(GL(1)^{k+s+t} \times SL(n') \times SL(m), \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t}\big) \otimes 1 + \big(\Lambda_1 \otimes \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} \otimes \Lambda_1\big) + 1 \otimes \big(\Lambda_1^* + \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s-1}\big) \qquad (s \geqslant 2)$$

with $n' = km + t - n$. Hence, it is the case dealt with in §4.1 when $kn' \leqslant m$ and it is the case (II-a) when $kn' \gneq m$. (2) Since the $GL(n)$ part of the generic isotropy subgroup of $(GL(1)^2 \times GL(n), \Lambda_1 + \Lambda_1^*)$ is

$$\left\{\left(\begin{array}{c|c} \alpha & 0 \\ \hline 0 & A \end{array}\right);\ \alpha \in GL(1),\ A \in GL(n-1)\right\},$$

(II-a) is isotropic P.V.-equivalent to

$$\Big(GL(1)^{k+s} \times GL(m) \times GL(n-1), \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k+s}\big) \otimes 1 + \big(\Lambda_1 \otimes \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} \otimes \Lambda_1\big) + 1 \otimes \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-2}\big)\Big)$$

(cf. Lemma 2.20), which is castling-equivalent to

$$\Big(GL(1)^{k+s} \times GL(n') \times GL(m), \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-2}\big) \otimes 1 + \big(\Lambda_1 \otimes \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k} \otimes \Lambda_1\big) + 1 \otimes \big(\overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{k+s}\big)\Big) \qquad (t \geqslant 2)$$

with $n' = km + t - n - 1$. Note that the scalar multiplications are no more independent. Assume that $kn' \leqslant m$. If it is a P.V., then $k + s \leqslant m$ by (4.3) since we have $k \geqslant 2$ by $km \gneq n \gneq m$. Hence, by Theorems 1.14 and 1.16, it is P.V.-equivalent to

$$\Big(GL(1)^{k+s} \times GL(n'), \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-2} + \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}\big) \otimes \big(\overbrace{1 + \cdots + 1}^{k+s}\big)\Big)$$

and we have $t - 2 + k(k+s) \leqslant n'$, i.e., (i). Next assume that $kn' \gneq m$. If it is a P.V., we have $n' \lneq m$ (see the proof of Theorem 4.15). Similarly as Theorem 4.13, by using Lemmas 4.10 and 4.11, we have our result. Note that, by Lemma 4.10, the statement in Theorem 4.13 holds even when the scalar multiplications are not independent. Q.E.D.

THEOREM 4.17. (1) *The case* (III-b) *in Lemma* 4.14 *is transformed to the case in* §4.1, *or to* (III-a) *by a castling transformation*. (2) *Consider the case* (III-a), *i.e.*,

$$\Big(GL(1)^{k+s+t} \times SL(m) \times SL(n), \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s}\big) \otimes 1 + \big(\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\big) + 1 \otimes \big(\Lambda_1 + \overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t-1}\big)\Big)$$
$$(t \geqslant 3,\ km \gneq n \gneq m).$$

This is a P.V. if and only if $(k, m, n-1) \in R$ *and* $(s+k)a_{j+1} + (t-2)a_{j+2} \leqslant a_{j+1}m - a_j(n-1)$ *for* $j = \psi(k, m, n-1)$ (*see Definition* 4.12).

PROOF. (1) is obvious. Similarly as Theorem 4.16, (III-a) is isotropic P.V.-equivalent to

$$\Big(GL(1)^{k+s} \times GL(m) \times GL(n-1), \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k+s}\big) \otimes 1 + (\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1) + 1 \otimes \big(\Lambda_1^* + \cdots + \Lambda_1^*\big)\Big).$$

If it is a P.V., then we have $m \lneq n-1$ by Theorem 4.5. Thus we have (2) similarly as Theorem 4.16. Q.E.D.

THEOREM 4.18. *Consider the case* (IV) *in Lemma* 4.14, *i.e.*,

$$\Big(GL(1)^{k+s+t} \times SL(m) \times SL(n), \big(\Lambda_1^* + \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s-1}\big) \otimes 1 + \big(\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1\big) + 1 \otimes \big(\Lambda_1^* + \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t-1}\big)\Big)$$
$$(s \geqslant 1,\ t \geqslant 1,\ km \gneq n \gneq m).$$

This is P.V.-equivalent to

$$\Big(GL(m-1) \times GL(n-1), \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k+s-2}\big) \otimes 1 + \big(\overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}\big) + 1 \otimes \big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k+t-2}\big)\Big).$$

Put $n' = km + t - n - 1$. *Then it is a P.V. if and only if* (1) $kn' \leqslant m-1$ *and* $k + t - 2 + k(k+s-2) \leqslant n'$, *or* (2) $kn' \gneq m-1$, $(k, n', m-1) \in R$ *and* $(k+t-2)a_{j+1} + (k+s-2)a_{j+2} \leqslant a_{j+1}n' - a_j(m-1)$ *for* $j = \psi(k, n', m-1)$ (*see Definition* 4.12).

PROOF. By considering the generic isotropy subalgebra of $(GL(1)^3 \times GL(m) \times GL(n),\ \Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$ similarly as the proof of Lemma 2.20, we obtain the former statement. By a castling transformation, we have

$$\Big(GL(n')\times GL(m-1),\ \big(\overbrace{\Lambda_1+\cdots+\Lambda_1}^{k+t-2}\big)\otimes 1 + (\Lambda_1\otimes\Lambda_1+\cdots+\Lambda_1\otimes\Lambda_1) + 1\otimes\big(\overbrace{\Lambda_1^*+\cdots+\Lambda_1^*}^{k+s-2}\big)\Big)$$

with $n' = km + t - n - 1$. If $kn' \leqslant m - 1$, then we have $k + s - 2 \leqslant m - 1$ by (4.3). Hence, by Theorems 1.14 and 1.16, it is P.V.-equivalent to

$$\Big(GL(n'),\ \big(\overbrace{\Lambda_1+\cdots+\Lambda_1}^{k+t-2}\big)+\big(\overbrace{\Lambda_1+\cdots+\Lambda_1}^{k}\big)\otimes\big(\overbrace{1+\cdots+1}^{k+s-2}\big)\Big)$$

and we have (1). If $kn' \gneqq m - 1$, then we have (2) by Lemma 4.10 similarly as Theorem 4.13. Q.E.D.

5. List of (indecomposable) 2-simple P.V.'s of Type II.

5.1. *2-simple P.V.'s of type* II *obtained from any given simple P.V.* $(GL(1)^l \times G, \rho_1 + \cdots + \rho_l)$ *with* $\deg \rho_i \geqslant 2$ $(i = 1, \ldots, l)$.

(1)

$$\big(GL(1)^{l+r}\times G\times SL(n),\ (\sigma_1+\cdots+\sigma_r)\otimes\Lambda_1+(\rho_1+\cdots+\rho_l)\otimes 1\big)$$

for any representation $\sigma_1 + \cdots + \sigma_r$ of G and any natural number n satisfying $n \geqslant \deg \sigma_1 + \cdots + \deg \sigma_r$.

(2)

$$\Big(GL(1)^{l+t}\times G\times SL\big(\textstyle\sum_{i=1}^{k}\deg\rho_i+t-1\big),\ (\rho_1+\cdots+\rho_k)\otimes\Lambda_1+\big(\rho_{k+1}^*+\cdots+\rho_l^*\big)\otimes 1+1\otimes\big(\overbrace{\Lambda_1+\cdots+\Lambda_1}^{t}\big)\Big)\qquad(1\leqslant k\leqslant l)$$

for any $t \geqslant 0$.

(3)

$$\Big(GL(1)^{l+t}\times G\times SL(n),\ (\rho_1+\cdots+\rho_k)\otimes\Lambda_1+(\rho_{k+1}+\cdots+\rho_l)\otimes 1+1\otimes\big(\overbrace{\Lambda_1+\cdots+\Lambda_1}^{t-1}+\Lambda_1^*\big)\Big)\qquad(1\leqslant k\leqslant l)$$

for any pair of natural numbers (t, n) satisfying $t \geqslant 1$ and $n \geqslant t - 1 + \deg \rho_1 + \cdots + \deg \rho_k$.

Note that any one of (1)–(3) is P.V.-equivalent to a simple P.V. $(GL(1)^l \times G, \rho_1 + \cdots + \rho_l)$. For the convenience of the reader, we give the list of simple P.V.'s.

List of Simple P.V.'s $\big(GL(1)^l \times G, \rho_1 + \cdots + \rho_l\big)$.

(I) *Irreducible case* (see [**1**]).
(I-1) $G = SL(n)$, $\rho_1 = \Lambda_1, \Lambda_2, 2\Lambda_1$; $3\Lambda_1$ $(n = 2)$; Λ_3 $(n = 6, 7, 8)$.
(I-2) $G = Sp(n)$, $\rho_1 = \Lambda_1$; Λ_3 $(n = 3)$.
(I-3) $(GL(1) \times SO(n), \Lambda_1)$.

(I-4) ($GL(1) \times \mathrm{Spin}(n)$, (half-)spin rep.) with $n = 7, 9, 10, 11, 12, 14$.

(I-5) ($GL(1) \times G_2$, Λ_2, $V(7)$), ($GL(1) \times E_6$, Λ_1, $V(27)$), ($GL(1) \times E_7$, Λ_6, $V(56)$).

(II) *Nonirreducible case* ([**2**], with a correction in [**3**]).

(II-1) $G = SL(n)$;

$$\Lambda_1 \overbrace{+ \cdots +}^{l-1} \Lambda_1 + \Lambda_1^{(*)} \qquad (2 \leqslant l \leqslant n+1, n \geqslant 2),$$

$$2\Lambda_1 + \Lambda_1^{(*)},\ \Lambda_2 + \Lambda_1^{(*)} \overbrace{+ \cdots +}^{l} \Lambda_1^{(*)} \qquad (2 \leqslant l \leqslant 4, n \geqslant 4)$$

except $\Lambda_2 + \Lambda_1 + \Lambda_1 + \Lambda_1^*$ for $n =$ odd, $\Lambda_2 + \Lambda_2$ ($n =$ odd), $\Lambda_2 + \Lambda_2 + \Lambda_1^*$ ($n = 5$), $\Lambda_3 + \Lambda_1^{(*)}$ ($n = 6, 7$), $\Lambda_3 + \Lambda_1 + \Lambda_1$ ($n = 6$).

(II-2) $G = Sp(n)$; $\Lambda_1 + \Lambda_1$, $\Lambda_1 + \Lambda_1 + \Lambda_1$; $\Lambda_2 + \Lambda_1$ ($n = 2$); $\Lambda_3 + \Lambda_1$ ($n = 3$).

(II-3) $G = \mathrm{Spin}(n)$; spin rep. + vector rep. ($n = 7$), half-spin rep. + vector rep. ($n = 8, 10, 12$), $\Lambda + \Lambda$ ($n = 10$) where $\Lambda =$ the even half-spin rep.

Here Λ_1^* denotes the dual of Λ_1 and $\Lambda_1^{(*)}$ stands for Λ_1 or Λ_1^*. Note that $(G, \rho, V) \cong (G, \rho^*, V^*)$ as triplets when G is reductive. Since (Spin($2n$), even half-spin rep.) $\cong$ (Spin($2n$), odd half-spin rep.), we simply write as (Spin($2n$), half-spin rep.). Note also that $(Sp(2), \Lambda_2) \cong (SO(5), \Lambda_1)$.

5.2. 2-*simple P.V.'s of Type* II *dealt with in* §2, *i.e.*, $(GL(1)^{k+s+t} \times G \times SL(n),\ (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t))$ *with* $2 \leqslant \deg \rho_i \leqslant n$ ($i = 1, \ldots, k$) *and* $(\tau_1 + \cdots + \tau_t) \neq (\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$.

(4) $G = SL(m)$ ($2 \leqslant m < n$).

(4-I) $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes 2\Lambda_1^{(*)}(+\Lambda_1^{(*)} \otimes 1)$; $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^{(*)}(+T)$ with $T = \Lambda_1^{(*)} \otimes 1$, $(\Lambda_1^{(*)} + \Lambda_1^{(*)}) \otimes 1$, $1 \otimes \Lambda_1^{(*)}$, $\Lambda_1^{(*)} \otimes 1 + 1 \otimes \Lambda_1^{(*)}$.

(4-II) $n =$ even; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^{(*)} + T$.

(II-0) $T = (\Lambda_1 + \Lambda_1)^{(*)} \otimes 1 + 1 \otimes \Lambda_1^{(*)}$, $(\Lambda_1 + \Lambda_1^* + \Lambda_1^*) \otimes 1$, $(\Lambda_1 + \Lambda_1 + \Lambda_1)^{(*)} \otimes 1$.

(II-1) $m =$ even; $T = (\Lambda_1 + \Lambda_1 + \Lambda_1^*) \otimes 1$, $(\Lambda_1 + \Lambda_1^*) \otimes 1 + 1 \otimes \Lambda_1^{(*)}$.

(II-2) $m =$ odd; $T = \Lambda_2 \otimes 1$, $1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)})$.

(4-III) $n =$ odd.

(A) $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + T$ with:

(III-0-A) $T = (\Lambda_1 + \Lambda_1) \otimes 1 + 1 \otimes \Lambda_1^*$ ($m \geqslant 3$), $\Lambda_2^* \otimes 1$, $(\Lambda_1 + \Lambda_1 + \Lambda_1)^{(*)} \otimes 1$ ($m \geqslant 3$), $(\Lambda_1 + \Lambda_1)^{(*)} \otimes 1 + 1 \otimes \Lambda_1$.

(III-1-A) $m =$ even; $T = \Lambda_2 \otimes 1$, $(\Lambda_1 + \Lambda_1^*) \otimes 1 + 1 \otimes \Lambda_1$, $1 \otimes (\Lambda_1 + \Lambda_1^*)$, $1 \otimes (\Lambda_1^* + \Lambda_1^*)$.

(III-2-A) $m =$ odd; $T = (\Lambda_1 + \Lambda_1 + \Lambda_1^*) \otimes 1$, $1 \otimes (\Lambda_1 + \Lambda_1)$, $(\Lambda_1 + \Lambda_1^*) \otimes 1 + 1 \otimes \Lambda_1^*$.

(B) $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + T$ with:

(III-0-B) $T = (\Lambda_1 + \Lambda_1^* + \Lambda_1^*) \otimes 1$, $(\Lambda_1 + \Lambda_1 + \Lambda_1)^{(*)} \otimes 1$, $(\Lambda_1 + \Lambda_1)^{(*)} \otimes 1 + 1 \otimes \Lambda_1^{(*)}$.

(III-1-B) $m =$ even; $T = (\Lambda_1 + \Lambda_1 + \Lambda_1^*) \otimes 1$, $1 \otimes (\Lambda_1^* + \Lambda_1^*)$, $(\Lambda_1 + \Lambda_1^*) \otimes 1 + 1 \otimes \Lambda_1^{(*)}$.

(III-2-B) $m =$ odd; $T = \Lambda_2 \otimes 1$, $1 \otimes (\Lambda_1 + \Lambda_1)$, $1 \otimes (\Lambda_1 + \Lambda_1^*)$.

(5) $G = SL(2)$ $(2 < n)$.

(5-I) $\rho' = 2\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^{(*)}(+\Lambda_1 \otimes 1)$.

(5-II) $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + T$ with:

(II-0) $T = 2\Lambda_1 \otimes 1, 2\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1$.

(II-1) $n =$ even; $T = 3\Lambda_1 \otimes 1, (2\Lambda_1 + \Lambda_1) \otimes 1, 2\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^*$.

(5-III) $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + T$ with $T = 2\Lambda_1 \otimes 1$, $3\Lambda_1 \otimes 1$, $(2\Lambda_1 + \Lambda_1) \otimes 1$, $2\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)}$.

(5-IV) $n = 5$; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^* + \Lambda_2^*)$.

(5-V) $n = 6$; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3(+T)$ with $T = 1 \otimes \Lambda_1^*$, $\sigma \otimes 1$ ($\sigma = \Lambda_1, 2\Lambda_1, 3\Lambda_1, \Lambda_1 + \Lambda_1, 2\Lambda_1 + \Lambda_1, \Lambda_1 + \Lambda_1 + \Lambda_1$)

(5-VI) $n = 7$; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3^{(*)}(+\Lambda_1 \otimes 1)$.

(6) $G = SL(3)$ $(3 < n)$.

(6-I) $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^{(*)} + 2\Lambda_1^{(*)} \otimes 1$.

(6-II) $n = 5$; $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2)$.

(7) $G = SL(4)$ $(4 < n)$.

(7-I) $n =$ odd; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + T$ with $T = 2\Lambda_1 \otimes 1$, $(\Lambda_2 + \Lambda_1^{(*)}) \otimes 1$, $\Lambda_2 \otimes 1 + 1 \otimes \Lambda_1^*$.

(7-II) $n = 5$; $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2)$.

(7-III) $n = 6$; $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3(+T)$ with $T = \Lambda_1^* \otimes 1$, $\Lambda_2^{(*)} \otimes 1$, $(\Lambda_1^* + \Lambda_1^*) \otimes 1$, $1 \otimes \Lambda_1$.

(8) $G = SL(5)$ $(5 < n)$.

(8-I) $n =$ even; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^{(*)} + (\Lambda_2 + \Lambda_1^*) \otimes 1$.

(8-II) $n =$ odd; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + (\Lambda_2 + \Lambda_1^*) \otimes 1$, $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + T$ with $T = (\Lambda_2^* + \Lambda_1) \otimes 1$, $\Lambda_2^* \otimes 1 + 1 \otimes \Lambda_1^*$.

(8-III) $n = 6$; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3(+T)$ with $T = \Lambda_1^{(*)} \otimes 1$, $\Lambda_2^* \otimes 1$, $(\Lambda_1^* + \Lambda_1^{(*)}) \otimes 1$, $1 \otimes \Lambda_1^*$.

(8-IV) $n = 7$; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3^{(*)}$.

(9) $G = SL(2r)$ $(n = 2r + 1)$.

(9-I) $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + 1 \otimes (\Lambda_1 + \Lambda_1)$.

(9-II) $r = 3$; $\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3^{(*)}(+\Lambda_1^{(*)} \otimes 1)$ $(n = 7)$.

(10) $G = SL(n)$.

$$\rho' = \Lambda_1 \otimes \Lambda_1 + (\rho_1 + \cdots + \rho_k) \otimes 1 + 1 \otimes \left(\rho_{k+1}^* + \cdots + \rho_r^*\right)$$

where $(GL(1)^r \times SL(n), \rho_1 + \cdots + \rho_r)$ is a simple P.V.

(11) $G = Sp(m)$ $(2m < n)$.

(11-I) $n =$ odd; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2$.

(11-II) $n =$ odd; $m = 2$; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + T$ with $T = \Lambda_1 \otimes 1$, $\Lambda_2 \otimes 1$, $1 \otimes \Lambda_1^*$.

(11-III) $n = 6$, $m = 2$; $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_3$.

5.3. 2-*simple P.V.'s of Type* II *dealt with in* §3, *i.e.*,

$$\Big(GL(1)^{k+s+t} \times G \times SL(n),\ (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1$$
$$+ (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes \Big(\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{t}\Big)$$

with $2 \leqslant \deg \rho_i \leqslant n$ $(i = 1, \ldots, k)$ where $(G;\ \rho_1, \ldots, \rho_k;\ \sigma_1, \ldots, \sigma_s) \neq (SL(m);\ \Lambda_1, \ldots, \Lambda_1;\ \Lambda_1^{(*)}, \ldots, \Lambda_1^{(*)})$. First note that the P.V.'s in (3.8) (resp. (3.2), (3.9); (3.4), (3.5), (3.12)) belong to (1) (resp. (2); (3)) in §5.1.

(12) $(GL(1)^2 \times SL(4) \times SL(8),\ (\Lambda_2 + \Lambda_1) \otimes \Lambda_1)$.

(13) $G = SL(m)$.

(13-I) $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*) + T$ $(m \lneq n)$ *with* $T = \Lambda_2^{(*)} \otimes 1$, $(\Lambda_2^{(*)} + \Lambda_1) \otimes 1$, $(\Lambda_2^* + \Lambda_1^*) \otimes 1$, $\Lambda_2^{(*)} \otimes 1 + 1 \otimes \Lambda_1^*$, $\Lambda_2^* \otimes 1 + 1 \otimes \Lambda_1$, $(\Lambda_2 + \Lambda_1^*) \otimes 1$ $(m = \textit{even})$, $\Lambda_2 \otimes 1 + 1 \otimes \Lambda_1$ $(m = \textit{even})$, $\Lambda_3 \otimes 1$ $(m = 6)$, $\Lambda_3 \otimes 1 + 1 \otimes \Lambda_1$ $(m = 6)$.

(13-II) $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) + T$ $(n = m + 1)$ with $T = \Lambda_2 \otimes 1$, $\Lambda_2^* \otimes 1$ $(m = \text{even})$, $\Lambda_3 \otimes 1$ $(m = 6)$.

(13-III) $\rho' = \Lambda_2 \otimes \Lambda_1 + T$ $(n \geqslant \frac{1}{2}m(m-1))$ with $T = 1 \otimes (\Lambda_1^* + \Lambda_1^*)$ $(m = \text{odd})$, $1 \otimes (\Lambda_1 + \Lambda_1^* + \Lambda_1^*)$ $(m = \text{odd},\ n \geqslant \frac{1}{2}m(m-1) + 1)$, $\Lambda_1^* \otimes 1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*)$ $(m = 5)$, $1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1)$ $(m = 2m' + 1,\ n = 2m'^2 + m' + 1)$, $1 \otimes (\Lambda_1 + \Lambda_1)$ $(m = 2m' + 1,\ n = 2m'^2 + m')$, $\Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1)$ $(m = 5,\ n = 10)$.

(14) $G = Sp(m)$ $(n \geqslant 2m)$.

(14-I) $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*)(+T)$ with $T = \Lambda_1 \otimes 1$, $1 \otimes \Lambda_1^*$, $1 \otimes \Lambda_1$ $(n \geqslant 2m + 1)$.

(14-II) $\rho' = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1^{(*)}(+T)$ $(n = 2m)$ with $T = \Lambda_1 \otimes 1$, $1 \otimes \Lambda_1^{(*)}$.

(14-III) $(GL(1) \times Sp(m) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$.

(15) $G = \text{Spin}\ (10)$ $(n \geqslant 16)$, $\rho' = \text{half-spin rep.} \otimes \Lambda_1 + T$ with $T = 1 \otimes (\Lambda_1^* + \Lambda_1^*)$, $1 \otimes (\Lambda_1 + \Lambda_1^* + \Lambda_1^*)$ $(n \geqslant 17)$, $1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1)$ $(n = 17)$, $1 \otimes (\Lambda_1 + \Lambda_1)$ $(n = 16)$.

5.4. 2-*simple P.V.'s of Type* II *dealt with in* §4, *i.e.*,

$$\Big(GL(1)^{k+s_1+s_2+t_1+t_2} \times SL(m) \times SL(n),$$
$$\Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{s_1} + \overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{s_2}\Big) \otimes 1 + \overbrace{\Lambda_1 \otimes \Lambda_1 + \cdots + \Lambda_1 \otimes \Lambda_1}^{k}$$
$$+ 1 \otimes \Big(\overbrace{\Lambda_1 + \cdots + \Lambda_1}^{t_1} + \overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{t_2}\Big)\Big) \qquad (n \geqslant m \geqslant 2,\ k \geqslant 1).$$

First we shall give the list of P.V.'s when $n \geqslant km$. Let $N(m)$ be the set of pairs (a, b) of nonnegative integers satisfying $1 \leqslant a + b \leqslant m + 1$, and one of a and b is 0 or 1, i.e.,

$$\Big(GL(1)^{a+b} \times SL(m),\ \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{a} + \overbrace{\Lambda_1^* + \cdots + \Lambda_1^*}^{b}\Big)$$

is a P.V.

(16) The case when $n \geqslant km$.

First note that P.V.'s in (4.7) (resp. (4.8); (4.1)) belong to (1) (resp. (2); (3)) in §5.1.

(16-I) $n = m,\ k = 1;\ (s_1 + t_2, s_2 + t_1) \in N(m)$.

(16-II) $n = km$ $(k \geqslant 2)$.

(16-II-a) $t_1 = 0,\ 2 \leqslant t_2 \leqslant n;\ s_2 = 0,\ s_1 + kt_2 \leqslant m$.

(16-II-b) $2 \leqslant t_1 \leqslant n,\ t_2 = 0;\ s_1 = 0,\ s_2 + kt_1 \leqslant m$.

(16-III) $n = km + 1$, $t_1 \geqslant 3$; $t_2 = s_1 = 0$, $s_2 + k(t_1 - 1) \leqslant m$.

(16-IV) $n \geqslant km + t_1$, $n \gneq km$.

(16-IV-a) $k = 1$, $t_1 = 0$, $2 \leqslant t_2 \leqslant n$; $(s_1 + t_2, s_2) \in N(m)$.

(16-IV-b) $k \geqslant 2$, $t_1 = 0$, $2 \leqslant t_2 \leqslant n$; $s_2 = 0$, $s_1 + kt_2 \leqslant m$.

(16-IV-c) $k \geqslant 1$, $t_1 = 1, 2 \leqslant t_2 \leqslant n$; $s_2 = 0$, $s_1 + kt_2 \leqslant m$.

Next we shall give the list of P.V.'s when $km \gneq n$. In this case, we shall use the following notations: $j = \psi(k, m, n)$ for $(k, m, n) \in R$ (see Definition 4.12), and for $\{a_i\}$ defined by $a_{-1} = -1$, $a_0 = 0$, $a_i = ka_{i-1} - a_{i-2}$ $(i \geqslant 1)$ (see Lemma 4.6), put $b_i = a_i/a_{i+1}$.

(17) $s_1 = s_2 = t_1 = t_2 = 0$; $\dim G \geqslant \dim V$, i.e., $k + (m^2 - 1) + (n^2 - 1) \geqslant kmn$.

(18) $s_2 = t_2 = 0$.

(18-I) $m \geqslant kn'$, $s_1 = 0$; $t_1 \leqslant n' + 1$.

(18-II) $m \geqslant kn'$, $s_1 = 1$; $k + t_1 \leqslant n' + 1$.

(18-III) $m \geqslant kn'$, $2 \leqslant s_1 \leqslant m$; $t_1 + ks_1 \leqslant n'$.

(18-IV) $kn' \gneq m$, $s_1 \geqslant 1$; $t_1 + ks_1 \leqslant n' - b_j(m - s_1)$.

(18-V) $kn' \gneq m$, $n' \geqslant km'$, $s_1 = 0$, $t_1 = 1$; $k \leqslant m' + 1$.

(18-VI) $kn' \gneq m$, $n' \geqslant km'$, $s_1 = 0$, $2 \leqslant t_1 \leqslant n'$; $kt_1 \leqslant m'$.

(18-VII) $kn' \gneq m$, $km' \gneq n'$; $s_1 = 0$, $t_1 \geqslant 1$, $kt_1 \leqslant m' - b_j(n' - t_1)$.

Here, $n' = km + t_1 - n$ $(\lneq m)$, $m' = kn' - m$ $(\lneq n')$, and (k, n', m) (resp. $(k, m', n')) \in R$, $j = \psi(k, n', m)$ (resp. $\psi(k, m', n')$) in (18-IV) (resp. (18-VII)). Note that (18-I) (resp. (18-II), (18-V)) belong to (1) (resp. (3)) in §5.1.

(19) $t_2 = 0$, $s_2 = 1$, $s_1 \geqslant 2$.

(19-I) $m \geqslant kn'$; $t_1 + ks_1 \leqslant n'$.

(19-II) $kn' \gneq m$; $t_1 + ks_1 \leqslant n' - b_j(m - s_1)$ where $n' = km + t_1 - n$ $(\lneq m)$, $(k, n', m - 1) \in R$, $j = \psi(k, n', m - 1)$.

(20) $t_2 = 0$, $s_2 \geqslant 1$, $s_1 = 0$.

(20-I) $m \geqslant kn' + s_2$; $t_1 \leqslant n' + 1$.

(20-II) $m = kn' + s_2 - 1$; $t_1 = 0, 1$, $k + t_1 \leqslant n' + 1$.

(20-III) $m = kn' + 1$, $m \geqslant s_2 \geqslant 3$; $t_1 = 0$, $k(s_2 - 1) \leqslant n'$.

(20-IV) $m = kn'$, $m \geqslant s_2 \geqslant 2$; $t_1 = 0$, $ks_2 \leqslant n'$.

(20-V) $kn' \gneq m$, $n' \geqslant km'$, $t_1 = 0$; $s_2 \leqslant m' + 1$.

(20-VI) $kn' \gneq m$, $n' \geqslant km'$, $t_1 = 1$; $s_2 + kt_1 \leqslant m' + 1$.

(20-VII) $kn' \gneq m$, $n' \geqslant km'$, $n' \geqslant t_1 \geqslant 2$; $s_2 + kt_1 \leqslant m'$.

(20-VIII) $kn' \gneq m$, $km' \gneq n'$, $t_1 \geqslant 1$; $s_2 + kt_1 \leqslant m' - b_j(n' - t_1)$.

(20-IX) $kn' \gneq m$, $km' \gneq n'$, $m' \geqslant kn''$, $t_1 = 0$, $s_2 = 1$; $k \leqslant n'' + 1$.

(20-X) $kn' \gneq m$, $km' \gneq n'$, $m' \geqslant kn''$, $t_1 = 0$, $m' \geqslant s_2 \geqslant 2$; $ks_2 \leqslant n''$.

(20-XI) $kn' \gneq m$, $km' \gneq n'$, $kn'' \gneq m'$, $t_1 = 0$, $s_2 \geqslant 1$; $ks_2 \leqslant n'' - b_j(n'' - s_2)$.

Here $n' = km + t_1 - n$ $(\lneq m)$, $m' = kn' + s_2 - m$ $(\lneq n')$, $n'' = km' - n'$ $(\lneq m')$ and (k, m', n') (resp. $(k, n'', m')) \in R$, $j = \psi(k, m', n')$ (resp. $\psi(k, n'', m')$) in (20-VIII) (resp. (20-XI)).

Note that (20-I), (20-V) (resp. (20-II); (20-VI), (20-IX)) belong to (1) (resp. (2); (3)) in §5.1.

(21) $t_2 = 0$, $s_2 \geqslant 1$, $s_1 = 1$.

(21-I) $kn' \gneq m$, $n' \geqslant km'$; $(s_2 - 1) + k(k + t_1) \leqslant m'$.

(21-II) $kn' \gneq m$, $km' \gneq n'$; $(s_2 - 1) + k(k + t_1) \leqslant m' - b_j(m' - k - t_1)$.

Here $n' = km + t_1 - n$, $m' = kn' + s_2 - m$ and $(k, m', n') \in R$, $j = \psi(k, m', n')$.

(22) $t_2 = 1$, $s_2 = 0$, $t_1 \geqslant 1$.

(22-I) $m \geqslant kn'$, $(t_1 - 1) + k(k + s_1) \leqslant n'$.

(22-II) $kn' \gneqq m$, $(t_1 - 1) + k(k + s_1) \leqslant n' - b_j(m - k - s_1)$.

Here $n' = km + t_1 - n$, $(k, n', m) \in R$, $j = \psi(k, n', m)$.

(23) $t_2 = s_2 = 1$.

(23-I) $m - 1 \geqslant kn'$, $(k + t_1 - 1) + k(k + s_1 - 1) \leqslant n'$.

(23-II) $kn' \gneqq m - 1$, $(k + t_1 - 1) + k(k + s_1 - 1) \leqslant n' - b_j(n - k - s_1)$.

Here $n' = km + t_1 - n$, $(k, n', m - 1) \in R$, $j = \psi(k, n', m - 1)$.

(24) $t_2 \geqslant 1$, $s_2 = t_1 = 0$; $s_1 + kt_2 \leqslant m - b_j(n - t_2)$ where $(k, m, n) \in R$, $j = \psi(k, m, n)$.

(25) $t_2 \geqslant 2$, $s_2 = 0$, $t_1 = 1$; $(s_1 + k) + k(t_2 - 1) \leqslant m - b_j(n - t_2)$ where $(k, m, n - 1) \in R$, $j = \psi(k, m, n - 1)$.

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